Nielsen equalizer theory

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Given a set of maps: \( f_1, \ldots, f_k : X \to Y \), the equalizer set is

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\text{Eq}(f_1, \ldots, f_k) = \{ x \in X \mid f_1(x) = \cdots = f_k(x) \}
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The points where all the functions agree.
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**Proposition**

*When $X$ and $Y$ have the same dimension, given $f_1, \ldots, f_k : X \to Y$ with $k > 2$, we can change the maps by homotopy to be equalizer free.*
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Make $\text{Coin}(f_1, f_i)$ finite for each $i$, then arrange for these sets to be distinct.
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Then the coincidence sets $\text{Coin}(f_1, f_i)$ will be submanifolds of $X$, and it’s possible that their intersections would be essentially nonempty.
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We can compute the coincidence sets:

$\text{Coin}(f, g)$ is points $(x, y)$ with $3x + y = 0x + 2y \mod \mathbb{Z}^2$, which is the “line” $y = 3x \mod \mathbb{Z}^2$. 
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We have 10 isolated equalizer points.
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We’ll define a Nielsen number (easy matrix formula for tori), and in this case

$$N(f, g, h) = 10.$$
On Tuesday, Peter Wong suggested I have a look at:


They give a very general theory for maps $f: X \to Y$ and a subset $B \subset Y$, and a Nielsen theory for counting $\#f^{-1}(B)$.

In various special cases, in appropriate codimensional settings, this gives:

- Nielsen fixed point theory ($B = \Delta$)
- Root theory ($B = \text{pt}$)
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With apologies to Dobreńko and Kucharski, let’s continue.
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Let $F, G : X \to Y^{k-1}$ be maps (codimension 0) given by

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Then \( F, G \) are maps of manifolds of the same dimension, and

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*With these dimensions, the maps can be changed by homotopy so that* \( \text{Eq}(f_1, \ldots, f_k) \) *is finite.*
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\[ F = (f_1, \ldots, f_1) \quad G' = (f_2', \ldots, f_k') \]

with \( \text{Coin}(F, G') = \text{Eq}(f_1, f_2', \ldots, f_k') \) finite.
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Easy to show that this is homotopy invariant, and has appropriate other properties.
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    : \\
    df_1 - df_k
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Equivalently, $x, x' \in \text{Eq}(f_1, \ldots, f_k)$ are in the same class when

$$x, x' \in p \text{Eq}(\tilde{f}_1, \alpha_2 \tilde{f}_2, \ldots, \alpha_k \tilde{f}_k)$$

for $\alpha_i \in \pi_1(Y)$. 
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We say \((\alpha_2, \ldots, \alpha_k) \sim (\beta_2, \ldots, \beta_k)\) if and only if there is \(z \in \pi_1(X)\) with

\[
\beta_i = \varphi_1(z) \alpha_i \varphi_i(z)^{-1}
\]

for all \(i\).
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\[ x, x' \in \text{Eq}(f_1, \ldots, f_k) \] are in the same class when there is a path \( \gamma \) from \( x \) to \( x' \) with

\[ f_i(\gamma) \simeq f_1(\gamma) \text{ for all } i \]
A class is essential when its index (or semi-index) is nonzero, and the number of such classes is $N(f_1, \ldots, f_k)$. 

We also get $R(f_1, \ldots, f_k)$ and $L(f_1, \ldots, f_k)$ in the usual way. Also we have a "minimal equalizer number" with 

$$ME(f_1, \ldots, f_k) \leq N(f_1, \ldots, f_k),$$

and these are equal when $(k-1)n \neq 2$.

For more than 2 maps, this always holds except for 3 maps on dimensions $2 \to 1$. 

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**Theorem**

*If* $Y$ *is a Jiang space, then all nonempty equalizer classes have the same index.*
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**Theorem**

*If $f_1, \ldots, f_k : T^{(k-1)n} \to T^n$ by matrices $A_1, \ldots, A_k$, then*

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Our old example: $f, g, h: T^2 \to S^1$ by

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Then we have

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For each \( i, j \) we have

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**Theorem**

*Any coincidence class containing an essential equalizer class must be geometrically essential.*
Our old example:

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\text{Coin}(f, g) \\
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Each one contains essential equalizer points. So yes, they are essential. So \(N(f, h) = 2\) in this case. Similarly \(N(f, g) = N(g, h) = 1\).
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So Jezierski *decreases* the domain dimension to get codimension 0.
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Jezierski, *The Nielsen coincidence number of maps into tori*, *Quaestiones Mathematicae*, 2001

gives a method for finding $N(f_1, f_2)$ by observing that they restrict to maps $T^2 \to T^2$, and this restriction respects the Nielsen number.

So Jezierski *decreases* the domain dimension to get codimension 0.

This only works because $T^2 \subset T^7$. 
Take two maps \( f_1, f_2 : T^7 \to T^2 \) with matrices \( A_1, A_2 \), and assume that \( A_2 - A_1 \) has rank 2.
Take two maps $f_1, f_2 : T^7 \to T^2$ with matrices $A_1, A_2$, and assume that $A_2 - A_1$ has rank 2.

We do the opposite:
Take two maps \( f_1, f_2 : T^7 \rightarrow T^2 \) with matrices \( A_1, A_2 \), and assume that \( A_2 - A_1 \) has rank 2.

We do the opposite:

*Increase* the domain dimension:
Take two maps $f_1, f_2 : T^7 \to T^2$ with matrices $A_1, A_2$, and assume that $A_2 - A_1$ has rank 2.

We do the opposite:

*Increase* the domain dimension: let $\overline{f}_1, \overline{f}_2 : T^8 \to T^2$ by adding columns of 0s to $A_1, A_2$. 
Take two maps $f_1, f_2 : T^7 \to T^2$ with matrices $A_1, A_2$, and assume that $A_2 - A_1$ has rank 2.

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Not hard to show that $N(f_1, f_2) = N(\bar{f}_1, \bar{f}_2)$. 
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Not hard to show that $N(f_1, f_2) = N(\overline{f}_1, \overline{f}_2)$.

Let $B_1, B_2$ be matrices of $\overline{f}_1, \overline{f}_2$, and $B_2 - B_1$ still has rank 2.
So we can invent matrices $B_3, \ldots B_5$ with

$$
\begin{bmatrix}
B_2 - B_1 \\
\vdots \\
B_5 - B_1
\end{bmatrix}
$$

of full rank (8).
So we can invent matrices $B_3, \ldots B_5$ with

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of full rank (8).

Thus $(\bar{f}_1, \bar{f}_2, g_3, \ldots, g_5)$ has essential equalizer classes and so $N(f_1, f_2) = N(\bar{f}_1, \bar{f}_2) \neq 0$
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Thus $(\overline{f}_1, \overline{f}_2, g_3, \ldots, g_5)$ has essential equalizer classes and so $N(f_1, f_2) = N(\overline{f}_1, \overline{f}_2) \neq 0$

Hopefully this trick can be used elsewhere when we need to prove that coincidence classes are essential.
Thank you!

Paper at arxiv: “Nielsen Equalizer Theory”, and in Topology and its applications