85 years of Nielsen theory: Coincidence Points

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Nielsen Theory and Related Topics 2013

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Today we'll do $f(x) = g(x)$ for two different maps.

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Then $N(f; X, A)$ is a lower bound for the number of fixed points of homotopic maps of pairs.

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No regularity assumptions about the number of images.

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(Better's talk)

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Try your own! But ask around first.

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Like in fixed point theory, we want an invariant to measure:

$$MC(f, g) = \min\{\#\,\mathrm{Coin}(f', g') \mid f' \simeq f,\ g' \simeq g\}$$

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Then $L(f,g)$ is the alternating sum of the traces of the composition:

$$H_q(X) \xrightarrow{f_{*q}} H_q(Y) \xrightarrow{D_Y} H^{n-q}(Y) \xrightarrow{g^{*n-q}} H^{n-q}(X) \xrightarrow{D_X^{-1}} H_q(X)$$

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This is homotopy invariant, and $L(f,g) \neq 0 \implies \text{Coin}(f,g) \neq \emptyset$.

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So we'll focus on pairs of orientable manifolds, same dimension.

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For the classes, they can be defined as coincidence sets of liftings like we did for fixed points.

Also a more geometric definition: $x, y \in \text{Coin}(f, g)$ are in the same class when there is a path α from x to y with $f(\alpha) \simeq g(\alpha)$.

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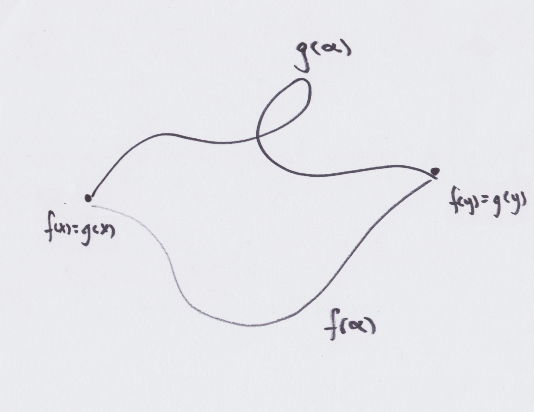

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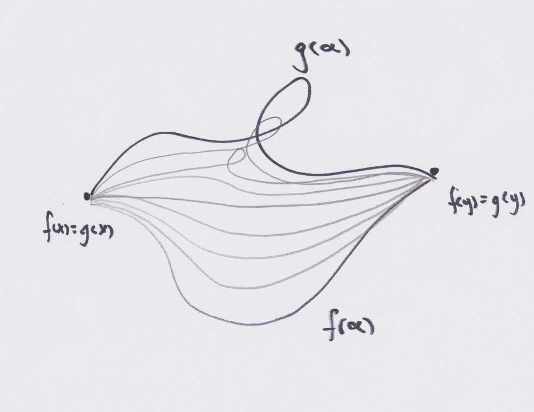

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$\mathcal{R}(f, g)$ is $\pi_1(Y)$ modulo "doubly-twisted conjugacy": $[\alpha] = [\beta]$ if and only if there is some $z \in \pi_1(X)$ with

$$\alpha = g_{\#}(z^{-1})\beta f_{\#}(z).$$

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Doubly-twisted conjugacy is again an interesting algebraic decision problem.

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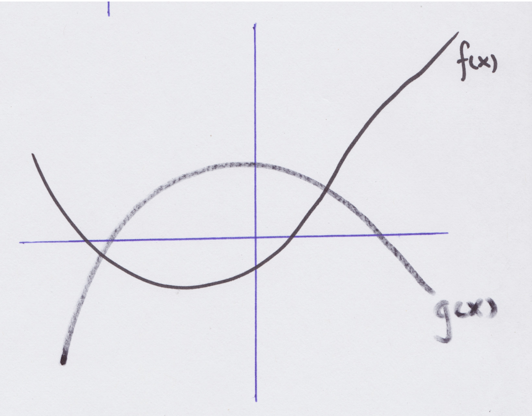

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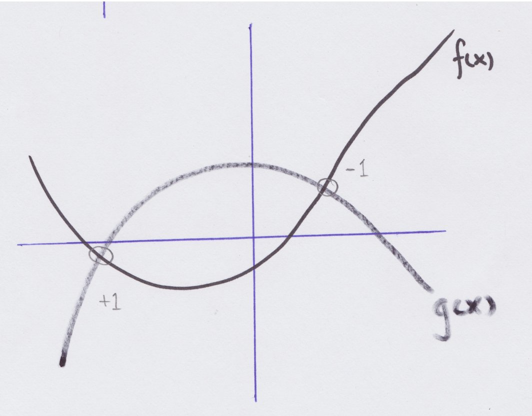

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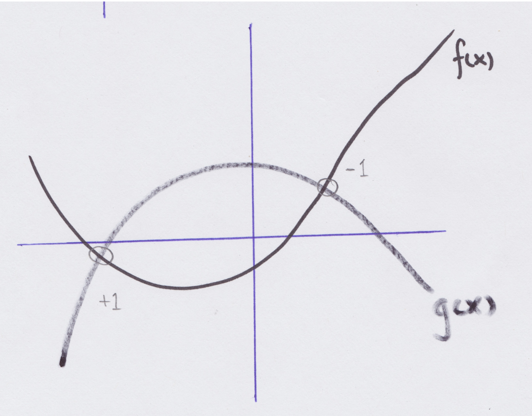

Again, the index is about the slopes when the intersect.

When the intersections are transverse, we can define the index at an isolated coincidence point as

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Homological definitions exist, and axiomatics.

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Theorem

When X and Y are orientable manifolds with $\dim X = \dim Y \neq 2$, *we have*

$$N(f,g) = MC(f,g)$$

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And

$$N(f, g) = |\det(B - A)|$$

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In Nielsen fixed point theory $f(x) = \text{id}(x)$, we change f by homotopies, but not id.

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Theorem

(Brooks) If the codomain is a manifold, then any coincidence set C achievable by changing both f and g can be obtained by changing only f.

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For such maps $N(f) \neq 0$ but $N(f, \mathrm{id}) = 0$.

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Similar issue in things like the Borsuk-Ulam question $f(x) = f(\tau(x))$, where homotopies of f result in specific (not arbitrary) homotopies of $f \circ \tau$.

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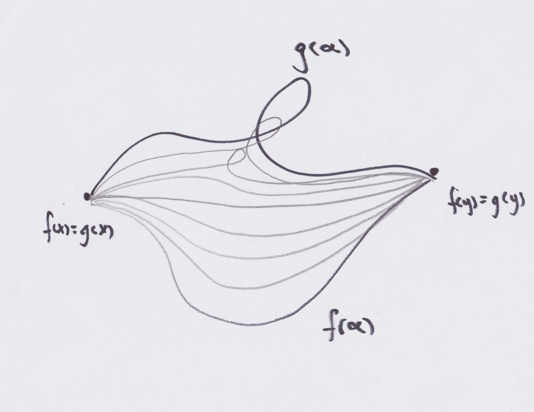
But on a nonorientable manifold there is some more subtlety.

$\downarrow$




x
α
y

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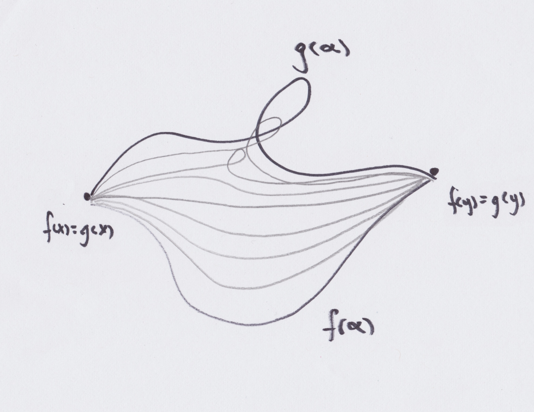
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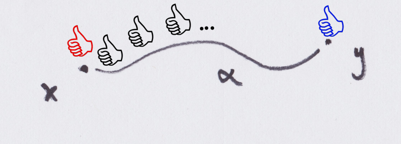

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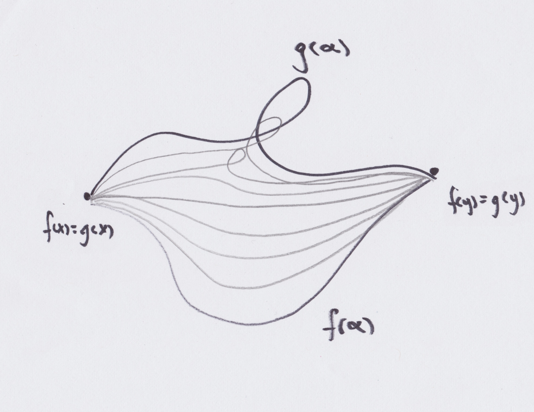

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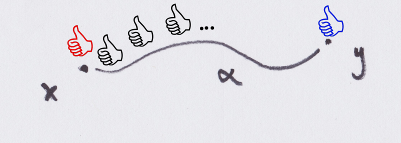
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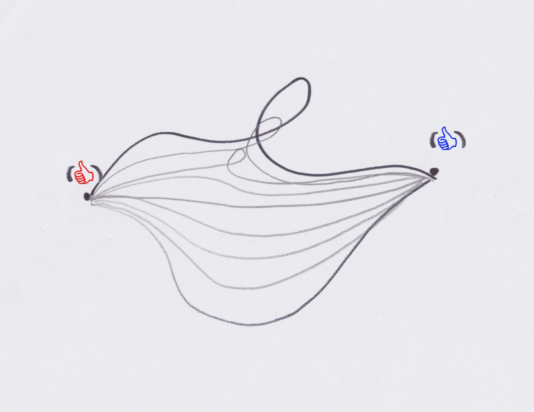

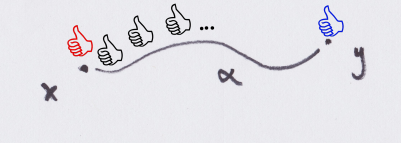
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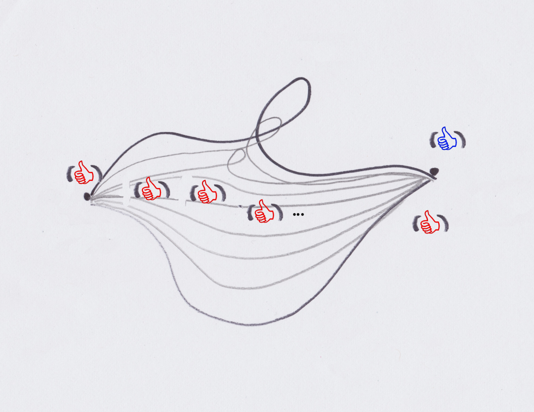
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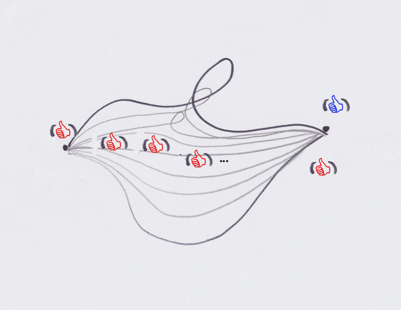

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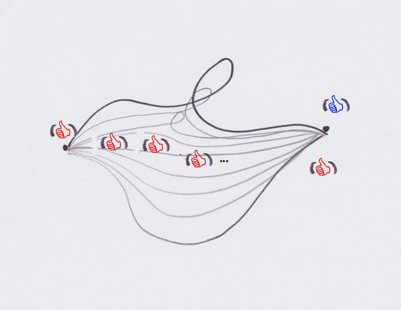

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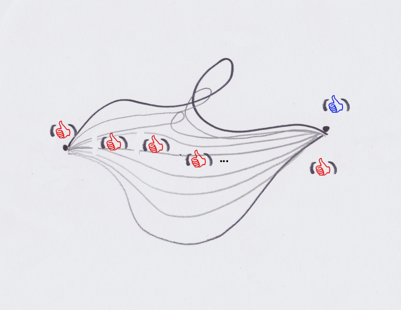

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OR,it can happen because the paths traverse orientation reversing loops.

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So the local index is not good enough. A mod 2 index would work, but this isn't very useful.

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Also D&J prove a Wecken theorem when $\dim \neq 2$.

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So Nielsen coincidence theory with different dimensions always focuses on the case $\dim X > \dim Y$.

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We'll want $N(f, g) \leq MC(f, g)$, but the latter will probably be infinite.

So we need to decide what exactly we're going to minimize.

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But actually there is more subtlety even here.

When the $MC(f, g)$ is finite, it may still be different from $MCC(f, g)$.

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$MC(f,g)$ and $MCC(f,g)$ cannot be simultaneously realized.

(But if one is zero, the other is too.)

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Maybe we need some other version of essentiality.

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Such a class is “geometrically essential”.

This can be very hard to compute.

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There is also a Jiang-type property for these spaces.

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If this class is nonzero, then the maps may not be made coincidence free.

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This turns out to be more manageable.

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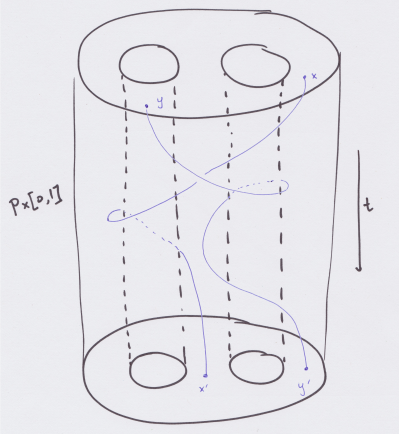
Another approach to all this is in terms of bordisms.

Recall in fixed point theory, the fixed point set varies during a homotopy like so:



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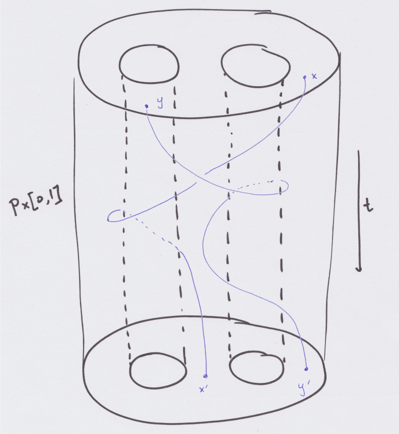
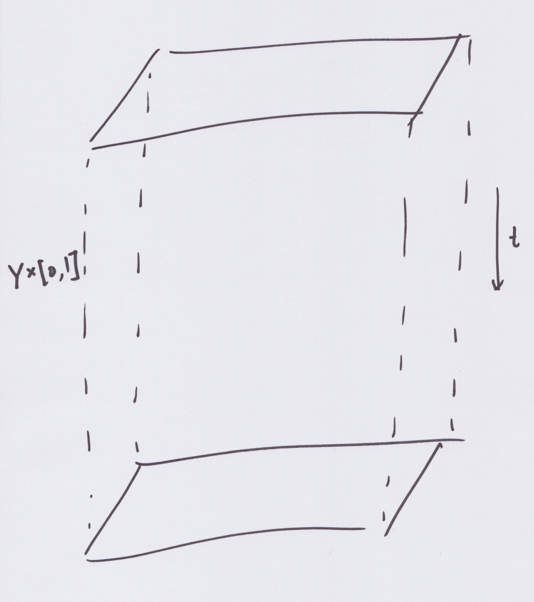


At each stage of the homotopy we have discrete points and integer invariants can be defined.

For positive codimension coincidence theory, the picture is like this:



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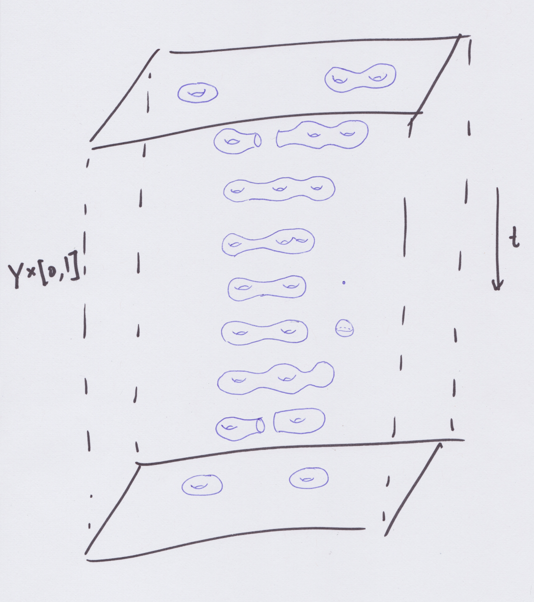

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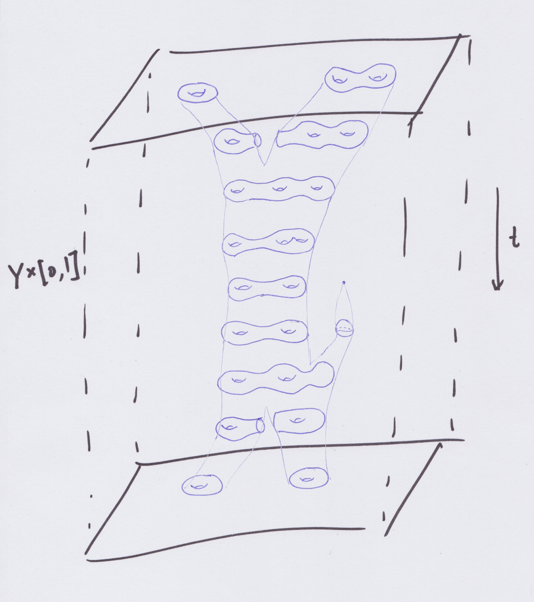

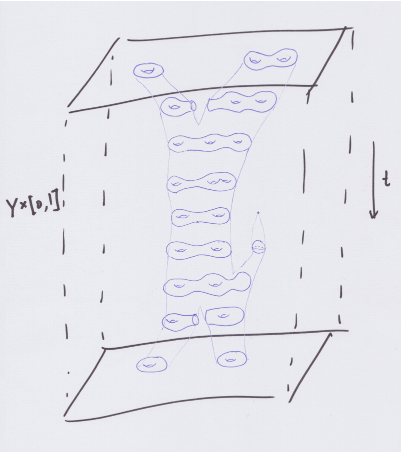

At each stage we have a submanifold which is cobordant with Coin(f, g) in a certain way.

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This is hard stuff, but obviously very deep. So it's worth it.

The end!

The end! (Finally!)