85 years of Nielsen theory: Fixed Points

P. Christopher Staecker

Fairfield University, Fairfield CT

Nielsen Theory and Related Topics 2013

Thanks

Thanks

Who my talk is for.

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Please ask questions.

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Some good books

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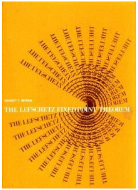
- 

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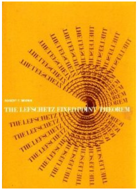
- 

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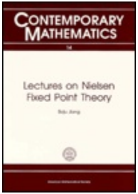
- 

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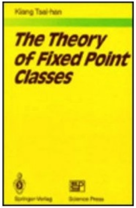
- 

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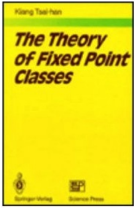
- 

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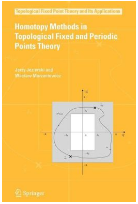
- 

Jerzy Jeziersky, Wacław Marzantowicz, Homotopy Methods in Topological Fixed and Periodic Points Theory, 2006

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This is a homotopy invariant, and it turns out is always an integer.

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All f_{*q} are zero except f_{*0} which is identity. So $L(f) = 1$, so Lefschetz's theorem implies Brouwer's.

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Then there is a nonzero trace f_{*q}, and so there is a simplex s with $f_q(s) = s$.

But s is topologically a q-disc, and so there is a fixed point in s by Brouwer.

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But this equals $L(f)$ by the Hopf Trace Theorem- the alternating sign is necessary to make this work.

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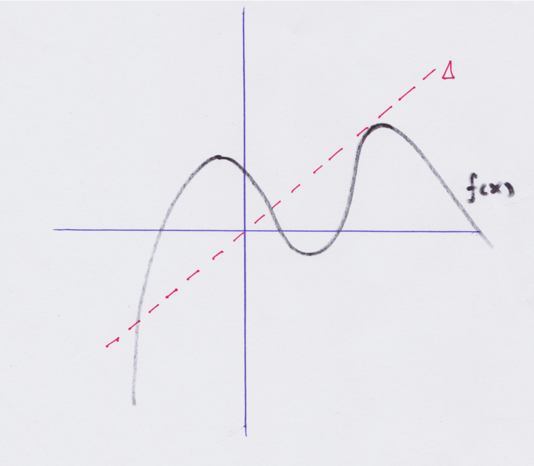
Fixed points are intersections of the graph of f and the diagonal Δ.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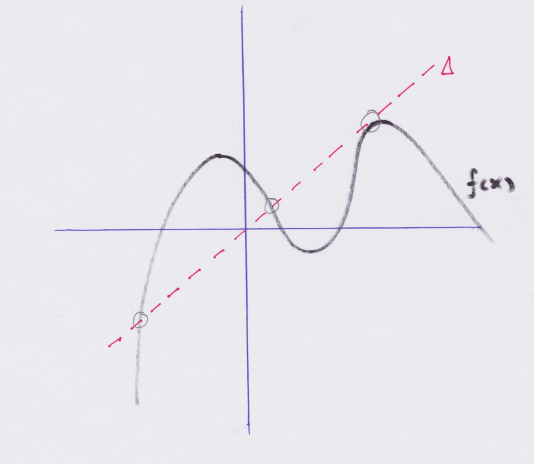
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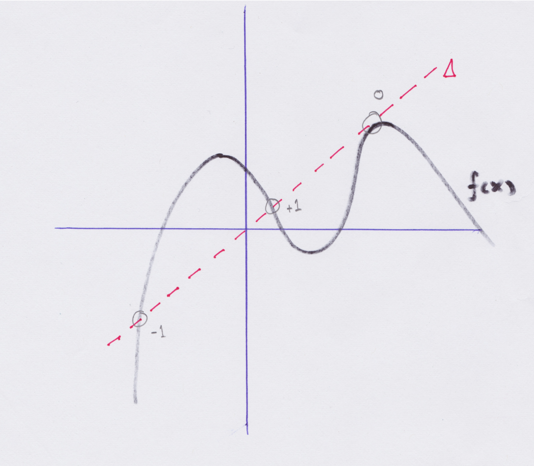
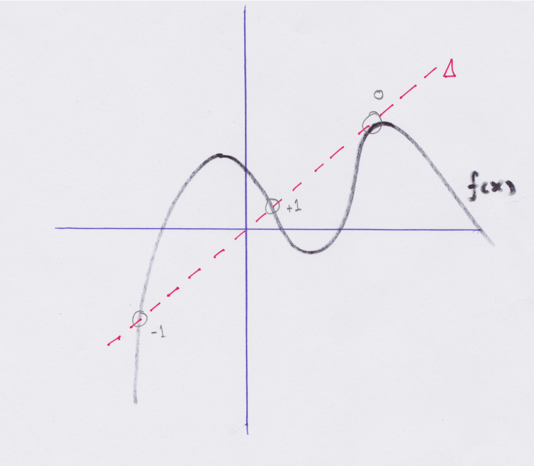


Δ
0
f(x)
+1
-1



The index depends on the slope as f passes through Δ.

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Axiomatic definitions exist too.

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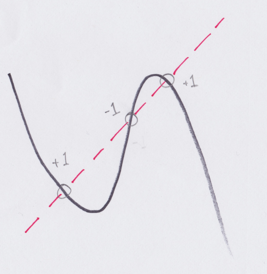

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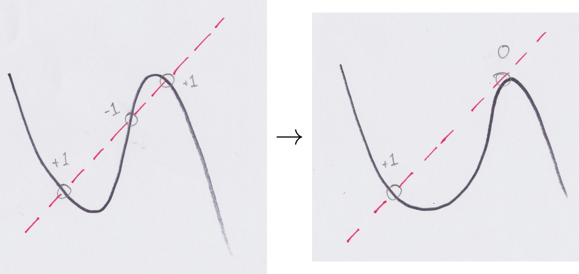

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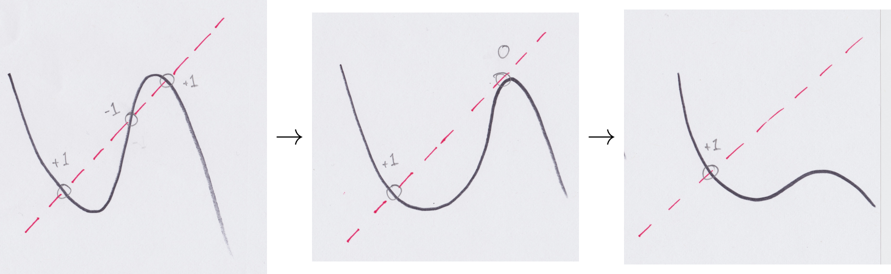

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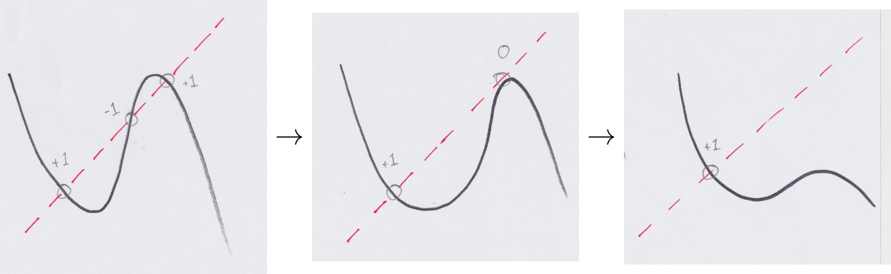

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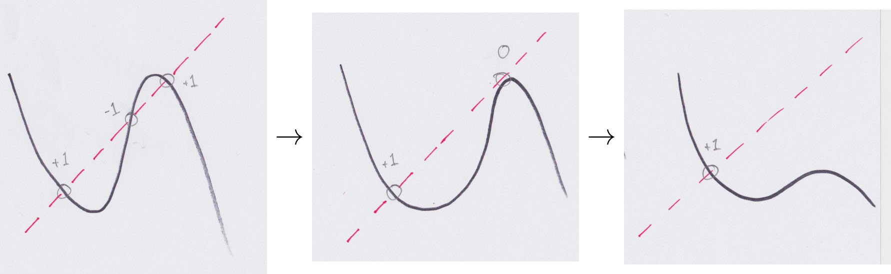

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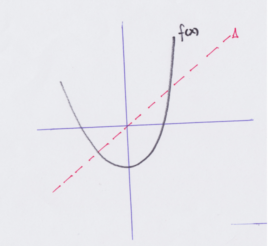

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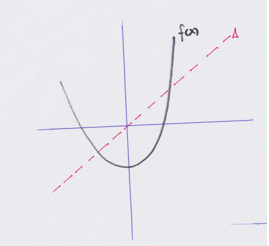

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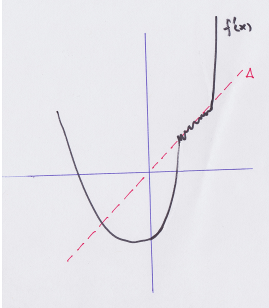

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Nielsen saw that this is a necessary condition for fixed points to be combined by a homotopy.

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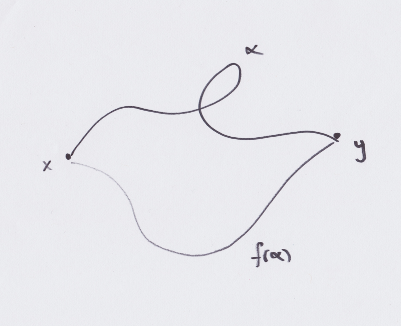

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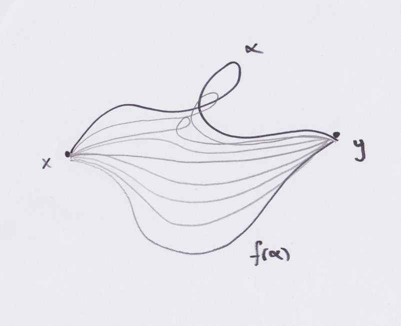

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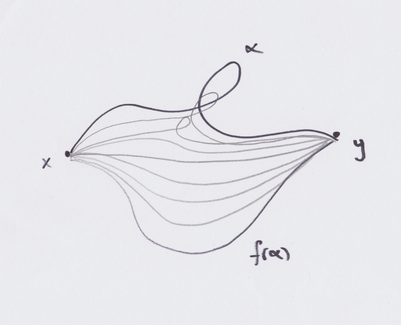

Pretty clear that this is necessary for x and y to be combined.

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Actually some sets $\mathrm{Fix}(\gamma\widetilde{f})$ may be empty, so really there's an inclusion:

$$\{ \text{ Fixed point classes } \} \hookrightarrow \{ \text{ Reidemeister classes } \}$$

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Lots of these become easier if we assume $f_\#$ is a group isomorphism.

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Automatically

$$N(f) \leq MF(f).$$

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These fixed points each have the same index ± 1, so $L(f) = \pm(1 - d)$

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So $[x] = [y]$ iff $x = y \mod (1-d)$.

So $\mathcal{R}(f) = \mathbb{Z}_{|1-d|}$.

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So we have $N(f) = |1-d|$, and also $MF(f) = |1-d|$ since $f(z) = z^d$ has $|1-d|$ fixed points.

So the Nielsen theory of the circle is easy.

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Tori Nielsen

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Further, these are all in different classes, and they all have the same index ± 1.

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Nilmanifolds allow a similar linearization of maps, and good formulas for Nielsen theory result. (Anosov, Fadell & Husseini 1985)

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Brown (1967) looked at this setting. When is there a product formula like

$$N(f) \stackrel{?}{=} N(\bar{f})N(f_b)$$

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See Heath's talk for more on fiber (fibre) methods.

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Unfortunately Jiang spaces all have π_1 abelian.

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Any space such that π_1 has R_∞ property cannot be a weakly Jiang space. (This isn't quite true)

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Then we can write $\widetilde{f}_q : C_q(\widetilde{X}) \to C_q(\widetilde{X})$ as a matrix with entries in $\mathbb{Z}\pi$, and we can do

$$\mathrm{tr}(\widetilde{f}_q : C_q(\widetilde{X}) \to C_q(\widetilde{X})) \in \mathbb{Z}\pi.$$

Reidemeister defined:

$$RT(\widetilde{f}) = \rho(\sum_q (-1)^q \operatorname{tr}(\widetilde{f}_q : C_q(\widetilde{X}) \to C_q(\widetilde{X})))$$

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Here $\rho : \mathbb{Z}\pi \to \mathbb{Z}\mathcal{R}(f)$ puts group elements into Reidemeister classes.

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Dimension 1 is easy, for dimension ≥ 3 there is enough "room" to deform $f(X)$ so that it intersects the diagonal Δ once for each essential class.

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What about surfaces?

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The paper is Fixed points and braids (1984 & 1985).

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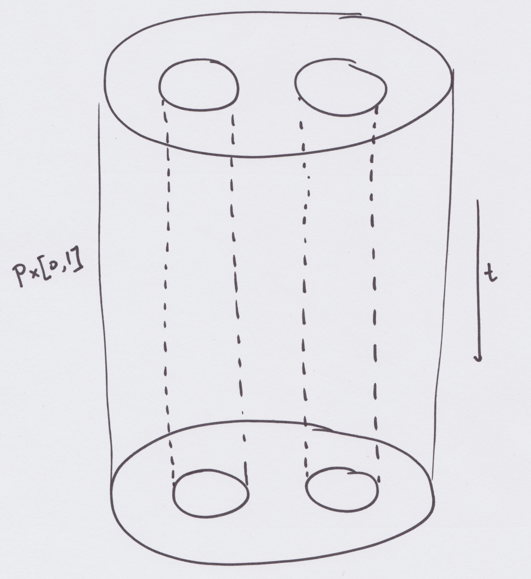
Let's use the pants surface P, and the homotopy itself is a map on $P \times [0,1]$.

This looks like:



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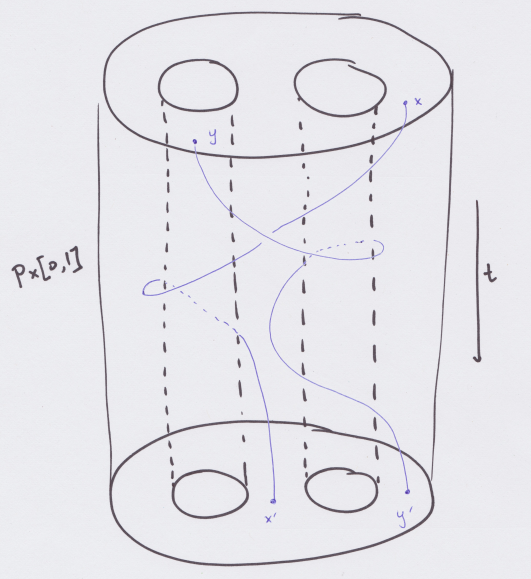

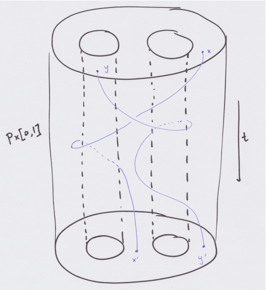

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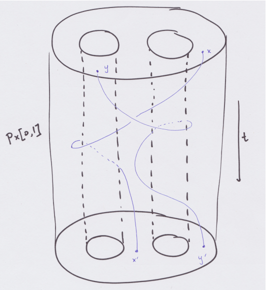

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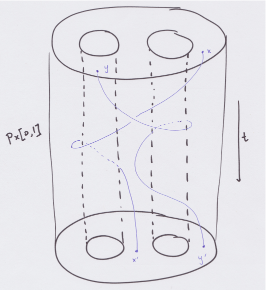

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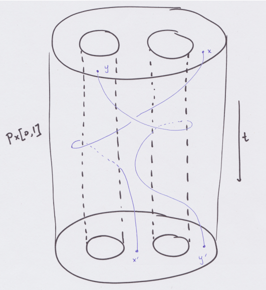
There is an algebraic theory for surface braids, using the "surface braid groups".



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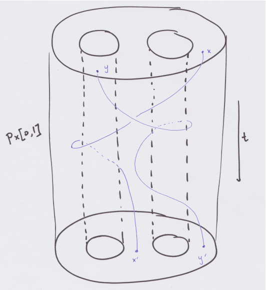

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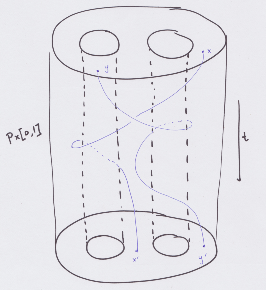

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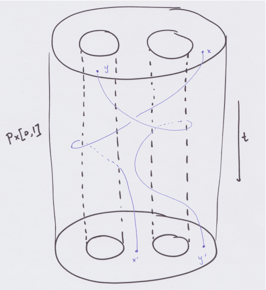

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Jiang shows that in his example, removing the two fixed points would require an algebraic formula to hold in the surface braid group.

Then he proves using the relations that this would be impossible.

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Several people asked whether $N(f)$ can be arbitrarily distant from $MF(f)$. Kelly showed that the difference can be arbitrarily large for any hyperbolic surface.

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Sometimes it does, sometimes it doesn't. (Khamsemanan's talk)

That's all for now!