85 years of Nielsen theory: Periodic Points

P. Christopher Staecker

Fairfield University, Fairfield CT

Nielsen Theory and Related Topics 2013

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This is all true, but not quite what we want.

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So $N(f^2) = 0$ even though all maps homotopic to f have at least 2 periodic points of period 2.

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In other words, when we look at $f^n(x) = x$, we should only be changing f by homotopy, not f^n.

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Even the sequence of Leftchetz numbers $(L(f), L(f^2), \dots)$ has a complicated structure.

This was studied by Dold (1983)

Theorem

(Dold) For each n, we have

$$\sum_{k|n} \mu(k) L(f^{n/k}) = 0 \mod n$$

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Lots more work on this followed.

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Jiang showed $\log N^\infty(f)$ is a lower bound for the topological entropy of f.

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There are two basic invariants, which J&M call the Nielsen-Jiang periodic numbers:

- $NP_n(f)$ counts the number of periodic points with minimal period n.
- $N\Phi_n(f)$ counts the number of all periodic points with period n.

We'll discuss the definitions of $NP_n(f)$ and $N\Phi_n(f)$, along with some basic properties and relations between them.

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This is true for nice spaces, but not always.

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It turns out this can be approached algebraically using the Reidemeister classes.

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So what we need is a map $\mathcal{R}(f^k) \to \mathcal{R}(f^n)$ which respects the periods correctly.

Here is the map $\iota_{k,n} : \mathcal{R}(f^k) \to \mathcal{R}(f^n)$, called the boost from level k to level n.

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The superscript in $[\alpha]^k$ just reminds us that this is the Reidemeister class of α in $\mathcal{R}(f^k)$.

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When $m \mid k \mid n$, we have

$$\iota_{m,n} = \iota_{k,n} \circ \iota_{m,k}$$

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Actually it's possible that every point in the orbit of x has the same Reidemeister class as x.

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We need to be careful about the orbits.

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For a class $[\alpha]^n \in \mathcal{R}(f^n)$, the Reidemeister orbit of α is:

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An orbit is reducible if it contains a reducible class, and essential if it contains an essential class.

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Once you show all of this is well-defined, it's not hard to show:

$$NP_n(f) \leq \min\{\#P_n(g) \mid g \simeq f\}$$

where P_n is the set of periodic points with minimal period n.

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$$\begin{aligned} NP_1(f) &= \# \text{ of essential irreducible classes of } \mathcal{R}(f^1) \\ &= \# \text{ of essential classes of } \mathcal{R}(f) \ = N(f) = |1-4| = 3. \end{aligned}$$

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So $NP_2(f) = 12$.

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So we have 3 reducible classes, so $NP_3(f) = 63 - 3 = 60$.

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So $NP_4(f) = 255 - 15 = 240$.

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We've repeatedly used the fact that the Reidemeister orbits at level n contain n distinct classes.

It's also true for tori that $\iota_{k,n}$ is injective (when $L(f^n) \neq 0$). Then we often don't need to compute the map $\iota_{k,n}$ exactly.

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Our earlier example: complex conjugate on S^1.

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The issue here is that the 2 points of period 2 are an inessential class in $\mathcal{R}(f^2)$, but they are preceded by an essential class in $\mathcal{R}(f)$.

So really we need to count those as being essential.

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Each such preceding essential orbit contributes to $N\Phi_n(f)$.

Specifically, a preceding essential orbit at level k should increase $N\Phi_n(f)$ by k.

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$$\bigcup_{k|n} \mathcal{OR}(f^k)$$

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Every orbit has a depth: the lowest level to which it reduces.

$\mathcal{G}$ is a minimal preceding system if its depth sum is minimal.

Then $N\Phi_n(f)$ is defined as:

$$N\Phi_n(f) = \sum_{O \in \mathcal{G}} d(O),$$

where d is the depth and $\mathcal{G}$ is any minimal preceding n-system.

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Pretty complicated!

If you're lucky, you'll be able to compute $N\Phi_n(f)$ by other means.

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Let's talk about this.

We want to relate a preceding n-system to the total number of all essential irreducible orbits.

Any preceding n-system automatically contains every essential irreducible orbit.

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So

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So half of our equality is always true. The other direction is not always true.

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So all classes at all levels reduce to level 1.

But the level 1 class is inessential because f is fixed point free.

All classes reduce except the bottom inessential one.

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Thus $N\Phi_n(f) = 1$ but $\sum_{k|n} NP_k(f) = 0$.

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To get the summation formula, we require that this never happens.

f is essentially reducible if the reduction of an essential class is essential.

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Theorem

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The essential reducibility condition holds for all maps on tori and all nil and solvmanifolds.

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For example,

$$\#P_6(f) = \#\,\mathrm{Fix}(f^6) - \#\,\mathrm{Fix}(f^3) - \#\,\mathrm{Fix}(f^2) + \#\,\mathrm{Fix}(f^1)$$

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For example,

$$\#P_6(f) = \#\,\mathrm{Fix}(f^6) - \#\,\mathrm{Fix}(f^3) - \#\,\mathrm{Fix}(f^2) + \#\,\mathrm{Fix}(f^1)$$

We "include" or "exclude" based on how exactly the levels divide one another.

This inclusion-exclusion idea actually works if we assume essential reducibility.

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Theorem

If f is essentially reducible, then

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This is obtained directly by Möbius inversion of the previous theorem.

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In particular this is true for tori.

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Probably we'll need to assume manifolds of dimension $\neq 2$.

For $N\Phi_n(f)$, the theorem we need is:

Theorem

If X is a manifold of dimension $\neq 2$ and f is a selfmap, then there is some map $g \simeq f$ with

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Realizing by a smooth map is different. (Jezierski's talk today)

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For $N\Phi_n$ this is an open question.

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Here $NP_n(f) = 0$ for all n, since the class at level 1 is inessential, and all classes at other levels reduce to it.

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Recall our example: the antipodal map on S^2.

Here $NP_n(f) = 0$ for all n, since the class at level 1 is inessential, and all classes at other levels reduce to it.

The simultaneous Wecken theorem would mean that all periodic points could be removed simultaneously.

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This point could be of minimal period 2, or it could be a fixed point.

In any case, we cannot remove every periodic point of f simultaneously.

That's all for now!

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Next time, coincidences.