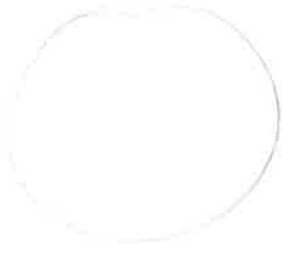
The rotation number for maps on graphs

With C. Bernhardt   CCNY 11/~~20~~21/13

TAMS

Classical : Poincaré

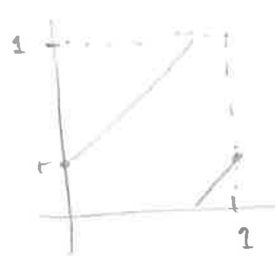
This is about homeos on circles :



The simplest are rotations.

If we view $S^1 = [0,1]/_{0=1}$,

a rotation looks like $f(x) = x + r \mod 1$



r is the fraction of the circle traversed by each point each time we do f.

the <u>rotation number</u>

So $r = 1/2$ means the pts go halfway around.

What if its not a rotation, but some other homeomorphism?

Here each pt goes a different fraction of the way around when we do f.

But what about under several iterations?

$x, f(x), f^2(x), f^3(x), \dots$

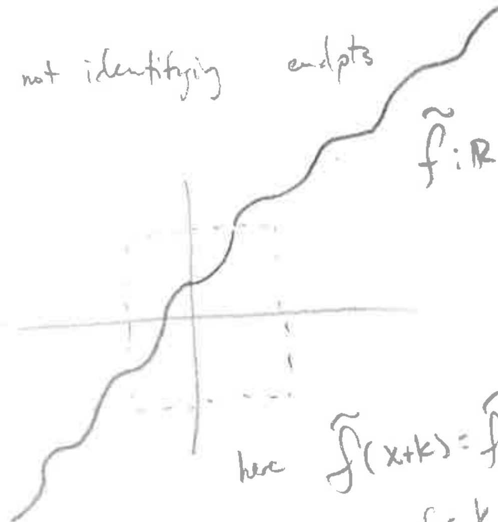
is there some kind of average rotation for the points in the sequence?

(Yes)

Poincaré's idea - look at the universal covering map

$$f : S^1 \to S^1$$

lift to $\tilde{f} : \mathbb{R} \to \mathbb{R}$ by not identifying endpts

So  $f : S^1 \to S^1$ becomes  $\tilde{f} : \mathbb{R} \to \mathbb{R}$

here $\tilde{f}(x+k) = \tilde{f}(x) + k$, for $k \in \mathbb{Z}$.

If we look at $\frac{\tilde{f}^n(x)}{n}$, this is basically an average of how many times x has gone around per iteration.

We use $\frac{\tilde{f}^n(x) - x}{n}$ just to clean things up a bit.

Poincaré showed that (when f is homeo) $\lim_{n\to\infty} \frac{\tilde{f}^n(x) - x}{n}$ always exists for any x,

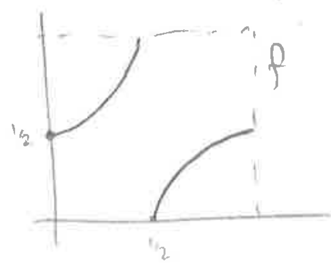
and This limit is the same for all pts.

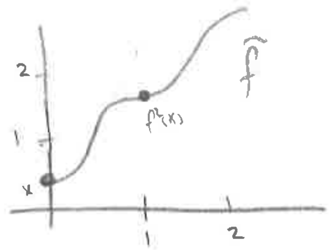
Called the rotation number of f, $\rho(f)$

~~Note - it's easy to compute $\rho(f)$ if f has periodic pts.~~

~~If $f^n(x) = x$ in S^1, then $\tilde{f}^n(\tilde{x}) - \tilde{x}$~~

Ex 1





Now $f(0) = \frac{1}{2}$ so $f^2(0) = 0$

$f^2(0) = f(1/2) = 0$

$\tilde{f}(0) = 1/2$ $\quad \tilde{f}^2(0) = 1$

$\tilde{f}(1/2) = 1$ $\quad \tilde{f}^3(0) = \frac{1}{2}+1$

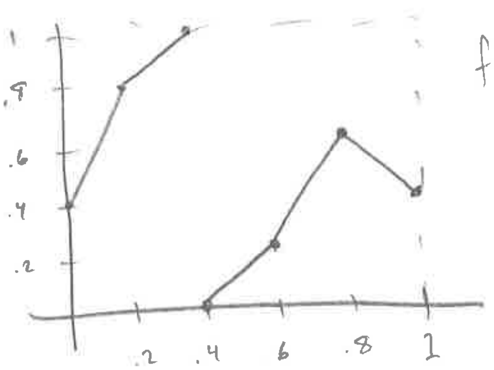
$\tilde{f}^4(0) = 2$

here, $\tilde{f}^{2k}(0) = k$

so $\lim_{n\to\infty} \frac{\tilde{f}^n(0) - 0}{n} = \lim_{n\to\infty} \frac{n/2}{n} = 1/2$

so $\rho(f) = 1/2$

Now: what about non-homeos?



Let's try the same thing:

turns out $f^2(0) = 0$, going around once

and $\rho_f(0) = 1/2$

but $f^3(.2) = .2$ going around twice

So $\rho_f(.2) = 2/3$

so for maps homotopic to identity,

$\lim_{n\to\infty} \frac{\tilde{f}^n(x) - x}{n}$ exists, but depends on x.

~~Thus~~ $\rho_f(x)$ varies from point to point, but these rotation numbers form an <u>interval</u>!

Our map has a pt with $\rho = 1/2$ and a pt with $\rho = 2/3$, so it must have pts with <u>all</u> reals in $[1/2, 2/3]$.

For example, $\exists$ a pt with rotation $3/5$:

Consider $A = [0, .2]$

$f(A) = [.4, .8]$

$f^2(A) = [0, .6]$

so $\exists\, B \subset A$ with $f^2(B) = [0, .2]$.

Then $f^2(f^2(B)) = [0, .6]$,

and $f^5(B)$ covers $[0, .2]$

so $\exists\, C \subset B \subset A$ with $f^5(C) = [0, .2]$

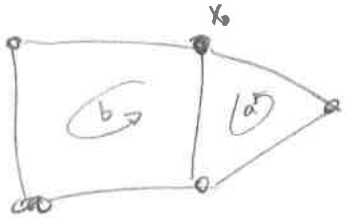
but $C \subset [0, .2]$, so $\exists$ a fixpt of f^5.

keep track of things, you'll see it went around 3 times,

so $\rho = 3/5$.

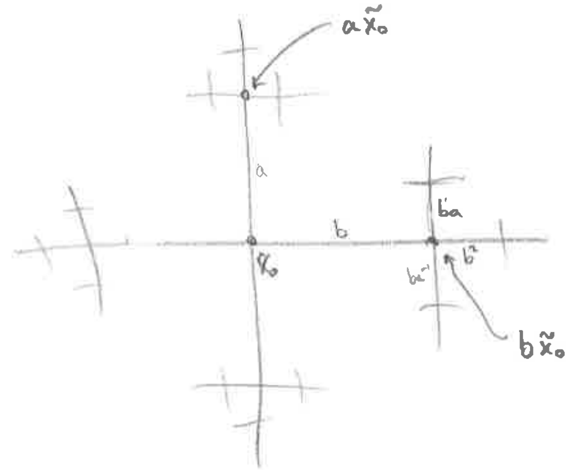
Similar arguments get other rationals. Irrationals are trickier.

Question: What about maps on graphs?

$X =$  $\circlearrowleft f$

We try the same stuff, but with fancier pictures.

The universal cover now is a big tree

$\tilde{X} =$ 

$\tilde{X}$ can be labeled according to $\pi_1(X)$, a free group

and $\tilde{f}(\gamma\tilde{x}) = f_*(\gamma) \cdot \tilde{f}(\tilde{x})$

but $f \simeq id$, so $\tilde{f}(\gamma\tilde{x}) = \gamma\tilde{f}(\tilde{x})$

If x is a periodic point, say $f^k(x) = x$,

then $\tilde{f}^k(\tilde{x}) = w\tilde{x}$ where $w \in \pi_1(X)$

w keeps track of how many times and in what order x "went around"

so $\tilde{f}^k(\tilde{x}) = aba^{-1}b^2\tilde{x}$ means

x went around a, then b, then a^{-1}, then b twice.

We define

$$\rho(x) = w^{1/k}$$

So $\rho(x) = (ab)^{1/2}$ means x on average goes halfway around ab each time.

$\rho(x)$ lives in a "free $\mathbb{Q}$-group"

exponents are formal, modulo $(w^n)^r = w^{nr}$ for $n \in \mathbb{Z}$.

So $(b^4)^{1/2} = b^2$

Quite a bit of details to check to show it's well-defined.

(choice of basepts, labelings, etc)

wrt these choices, it's only well-def up to conjugacy, which isn't surprising: rot around ab^2 is $\approx$ same as rot around b^2a

We show analogs of the "rotation interval"

<u>Thm</u> If $\exists\, x, y$ with $\rho(x) = w_1^{1/m}$, $\rho(y) = w_2^{1/n}$,

then $\exists\, z$ with $\rho(z) = v^{\frac{1}{rm+sn}}$

where $v = w_{i_1} w_{i_2} \cdots w_{i_{r+s}}$ $\quad i_j \in \{1, 2\}$

r of them are w_1,

s of them are w_2.

So the word can be any product of the two words, the exponent can be lots of different #s.

Our paper only defines $\rho(x)$ when x is a periodic pt.

Idea when x isn't a periodic pt.

look at the π_1-coord while we iterate:

$x \quad f(x) \quad f^2(x) \quad f^3(x) \quad \ldots$

$a \quad a \quad ab \quad abb \quad \ldots$

we get a sequence of words, typically growing.

If so, we can consider the "limit word" which is infinite

If this has the form $\underline{v\,w^{\infty}}$ for v, w finite,

then w is the rotation word, and keeping track of the growth of the seq tells us the exponent.

Not all points have rotation elements, and this exponent can be non-rational. So this element is in a "free $\mathbb{R}$-group"