Nielsen coincidence theory of iterates

(preliminary report)

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Nielsen theories

Nielsen fixed point theory studies

$$\mathrm{Fix}(f) = \{x \mid f(x) = x\}$$

in a homotopy-invariant way.

Generalizes to:

Coincidence theory:

$$\mathrm{Coin}(f, g) = \{x \mid f(x) = g(x)\}$$

Periodic points theory:

$$f^n(x) = x$$

Can study points with minimal period n, or points with any period n.

Why not?

Let's try both:

$$f^n(x) = g^n(x)$$

Or even:

$$f^n(x) = g^m(x)$$

For now, let's just stick with $n = m$.

NP_n and $N\Phi_n$

As in the Nielsen periodic point theory, we want Nielsen numbers $NP_n(f,g)$ and $N\Phi_n(f,g)$ which are meant to satisfy something like:

$$\begin{aligned} N\Phi_n(f,g) &\leq \min \#\{x \mid f^n(x) = g^n(x)\} \\ NP_n(f,g) &\leq \min \#\{x \mid f^n(x) = g^n(x) \text{ but } f^d(x) \neq f^d(x) \text{ for } d \mid n\} \end{aligned}$$

Call the set $\{x, f(x), f^2(x), \dots\}$ the *trajectory of x under f*. Then we are finding points x for which the trajectories under f and g meet at various iteration levels.

- $NP_n(f,g)$ should measure how many points have intersecting trajectories iterate n but not at any iterate $k \mid n$.
- $N\Phi_n(f,g)$ should measure how many points have intersecting trajectories at iterate n.

Setting

We consider compact manifolds without boundary.

Unlike in coincidence theory, we must require f and g to be selfmaps (so that we can iterate).

Periodicity and commutativity

To mimic the Nielsen periodic points theory, we want some kind of periodicity: like

$$f^k(x) = g^k(x) \quad \Rightarrow \quad f^n(x) = g^n(x) \qquad \text{for } k \mid n$$

This is not automatic.

If $f(x) = g(x)$, we get

$$f^2(x) = f(g(x)) \text{ and } g(f(x)) = g^2(x)$$

So we're going to need commutativity: $f \circ g = g \circ f$. (We'll be able to loosen this a bit.)

Ingredients

The Nielsen periodic points theory has 3 main ingredients:

- The fixed point index
- Reidemeister classes and boosts
- Reidemeister orbits

The index

We will use the coincidence index.

Coincidence points split into *coincidence classes*: $x, y \in \mathrm{Coin}(f, g)$ are in the same class when there is some path γ connecting them with $f(\gamma) \simeq g(\gamma)$.

A coincidence class $C \subset \mathrm{Coin}(f^n, g^n)$ is *essential* when $\mathrm{ind}(f^n, g^n, C)$ is nonzero.

Reidemeister classes

Every coincidence class has an associated *Reidemeister class* $[\alpha] \in \mathcal{R}(f^n, g^n)$, where

$$\mathcal{R}(f^n, g^n) = \pi_1(X)/\sim$$

with the relation

$$[\alpha] \sim [\beta] \iff \exists z \in \pi_1(X) \quad \beta = f_*^n(z)\alpha g_*^{-n}(z).$$

The boost

There is a *boost* function from one iteration level to another: for $k \mid n$, we have

$$\iota_{k,n} : \mathcal{R}(f^k, g^k) \to \mathcal{R}(f^n, g^n)$$

given by

$$\iota_{k,n}([\alpha]) = [f_*^{n-k}(\alpha)\, f_*^{n-2k}(g_*^k(\alpha)) \, \dots \, f_*^k(g_*^{n-2k}(\alpha))\, g_*^{n-k}(\alpha)]$$

This is well-defined on Reidemeister classes.
(Absolutely needs commutativity of f_* and g_*!)

NP_n

We say that a class $[\alpha] \in \mathcal{R}(f^n, g^n)$ is *irreducible* if it is not in the image of any boost.

Define:

$$NP_n(f, g) = \# \text{ of essential irreducible classes of } (f^n, g^n).$$

Note: no orbits! We have $f^n(x) = g^n(x) \Rightarrow f^{2n}(x) = g^{2n}(x)$.
Orbits require $f^n(x) = f^{2n}(x)$, which is something different entirely.

NP_n

NP_n as defined above satisfies:

- $NP_n(f,g)$ is homotopy invariant for both f and g.
- $NP_n(f,g) \leq \#\{x \mid f^n(x) = g^n(x), \quad f^{n/d}(x) \neq g^{n/d}(x)\}$
- $NP_n(f,\mathrm{id}) = NP_n(f)$ when X is essentially toral

NP_n does *not* actually require $f \circ g = g \circ f$, but only $f_* \circ g_* = g_* \circ f_*$.

This is a nicer requirement because it is preserved by homotopy.

The number of coincidences

From periodic points theory we have

$$\sum_{k|n} NP_k(f) \leq \min_{h \simeq f} \# \operatorname{Fix}(f^n)$$

This is not true simply replacing "Fix" with "Coin".

In fact

$$\min_{h \simeq f, l \simeq \mathrm{id}} \# \operatorname{Coin}(f^n, g^n)$$

isn't even a generalization.

One, rather than both

"On removing coincidences of two maps when only one, rather than both, of them may be deformed by a homotopy":

$$\min_{h\simeq f} \# \operatorname{Fix}(h) = \min_{h\simeq f, l\simeq \mathrm{id}} \# \operatorname{Coin}(h, l)$$

when the spaces are manifolds. (Brooks, 1971)

This does not hold for iterates.

That is, for $n \neq 1$ it is possible for

$$\min_{h\simeq f} \# \operatorname{Fix}(h^n) \neq \min_{h\simeq f, l\simeq \mathrm{id}} \# \operatorname{Coin}(h^n, l^n)$$

(Let $f : S^1 \to S^1$ be $f(z) = \bar{z}$, g a small rotation.)

Union of coincidences

The proper set of coincidences to look at is:

$$\bigcup_{k|n} \mathrm{Coin}(f^k, g^k)$$

This gives you $\mathrm{Fix}(f^n)$ when $g = \mathrm{id}$.

We can prove

$$\sum_{k|n} NP_k(f, g) \leq \# \bigcup_{k|n} \mathrm{Coin}(f^k, g^k)$$

(Still unsure whether

$$\min \# \mathrm{Fix}(f^n) = \min \# \bigcup_{k|n} \mathrm{Coin}(f^k, g^k)$$

for $g \simeq \mathrm{id}$. I don't think so.)

Two Nielsen numbers

Nielsen periodic point theory has two basic invariants:

$$NP_n(f) \leq \text{number of periodic points of } \textit{least} \text{ period } n$$
$$N\Phi_n(f) \leq \text{number of periodic points of period } n$$

The definition of the second one is a bit tricky- the trick is to make it a homotopy invariant of f (not f^n).

$N\Phi_n$

A set $\mathcal{G}$ of coincidence classes is a set of n-**representatives** if every essential class of (f^k, g^k) for $k \mid n$ reduces to something in $\mathcal{G}$.

The minimal size of a set of n-representatives is called $N\Phi_n(f, g)$.

Easy to see that

$$\sum_{k|n} NP_k(f, g) \leq N\Phi_n(f, g)$$

What we all want

Several theorems we would like to have about the sum of the NP_k:

If f, g are essentially reducible, then

$$N\Phi_n(f,g) = \sum_{k|n} NP_k(f,g).$$

With a few more conditions (Jiang, essentially reducible to the gcd, $N(f^n, g^n) \neq 0$),

$$N\Phi_n(f,g) = \sum_{k|n} NP_k(f,g) = N(f^n, g^n)$$

and

$$NP_n(f,g) = \sum_{\tau \subset \mathbf{p}(n)} (-1)^{\#\tau} N(f^{n:\tau}, g^{n:\tau})$$

Essential reduction to the gcd

Some of those will be easy, some are difficult. Some of these conditions are more complicated for coincidences.

With great effort, we have shown (using new methods)

Theorem

If $f, g : S^1 \to S^1$ have degrees a and b, then f, g essentially reduce to the gcd if and only if $\gcd(a, b) = 1$.

Hopefully something like this is true for tori and solvmanifolds, but our argument for circles doesn't generalize.

(Injective boosts ("essential torality") works fine for coincidences for circles and tori.)

Circles

On circles (and tori), all maps are essentially reducible, so we get

$$N\Phi_n(f,g) = \sum_{k|n} NP_k(f,g)$$

But not all maps on the circle are essentially reducible to the gcd, so we may expect

$$\sum_{k|n} NP_k(f,g) \neq N(f^n, g^n)$$

even when $N(f,g) \neq 0$.

An example

Let $f, g : S^1 \to S^1$ have degrees 0 and 2. Then the Reidemeister classes are:

$$\mathcal{R}(f, g) \cong \mathbb{Z}_2 \quad \mathcal{R}(f^2, g^2) \cong \mathbb{Z}_4 \quad \mathcal{R}(f^3, g^3) \cong \mathbb{Z}_8 \quad \mathcal{R}(f^6, g^6) \cong \mathbb{Z}_{64}.$$

The boosts are [multiplication by]

$$\iota_{1,2} = 2, \quad \iota_{1,3} = 4, \quad \iota_{1,6} = 32, \quad \iota_{2,6} = 16, \quad \iota_{3,6} = 8$$

$$(\iota_{2,6} = 0^4 + 0^2 2^2 + 2^4 = 16)$$

So $[16] \in \mathcal{R}(f^6, g^6)$ reduces to levels 3 and 2, but not 1.

$$\mathcal{R}(f,g) \cong \mathbb{Z}_2 \quad \mathcal{R}(f^2,g^2) \cong \mathbb{Z}_4 \quad \mathcal{R}(f^3,g^3) \cong \mathbb{Z}_8 \quad \mathcal{R}(f^6,g^6) \cong \mathbb{Z}_{64}.$$

$$\iota_{1,2} = 2, \quad \iota_{1,3} = 4, \quad \iota_{1,6} = 32, \quad \iota_{2,6} = 16, \quad \iota_{3,6} = 8$$

For this example, we can compute

$$\begin{aligned} NP_1(f,g) &= 2, \\ NP_2(f,g) &= 4 - 2 = 2, \\ NP_3(f,g) &= 8 - 2 = 6, \\ NP_6(f,g) &= 64 - 8 = 56 \end{aligned}$$

So we get

$$\sum_{k|6} NP_k(f,g) = 66 \quad \text{but} \quad N(f^6, g^6) = 64.$$

So

$$\sum_{k|6} NP_k(f,g) \neq N(f^6, g^6).$$

And the Möbius inversion fails too:

$$\sum_{\tau \subset \mathbf{p}(6)} (-1)^{\#\tau} N(f^{6:\tau}, g^{6:\tau}) = N(f,g) - N(f^2, g^2) - N(f^3, f^3) + N(f^6, g^6)$$

$$= 2 - 4 - 8 + 64 = 54$$

so

$$56 = NP_6(f,g) \neq \sum_{\tau \subset \mathbf{p}(6)} (-1)^{\#\tau} N(f^{6:\tau}, g^{6:\tau})$$

So essential reducibility to the gcd is necessary for these, even on the circle.

It's over!

Thanks!