Nielsen equalizer theory

P. Christopher Staecker

Fairfield University, Fairfield CT

Capitol Normal University, Beijing China, June 24, 2011

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Make $\text{Coin}(f_1, f_i)$ finite for each i, then arrange for these sets to be distinct.

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Then the coincidence sets $\text{Coin}(f_1, f_i)$ will be submanifolds of X, and it's possible that their intersections would be essentially nonempty.

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$\mathrm{Coin}(f, g)$ is points (x, y) with $3x + y = 0x + 2y \mod \mathbb{Z}^2$, which is the "line" $y = 3x \mod \mathbb{Z}^2$.

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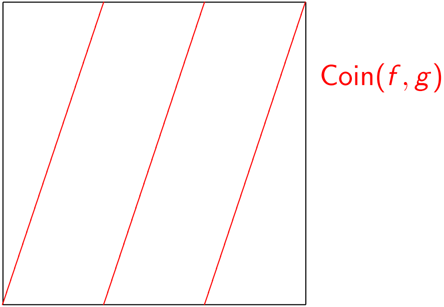

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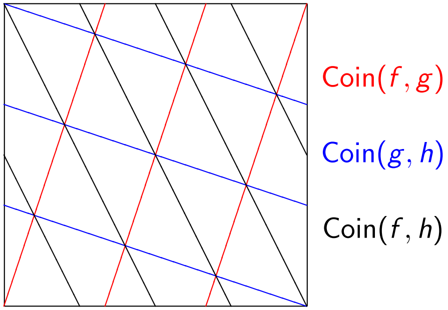

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We'll define a Nielsen number (easy matrix formula for tori), and in this case

$$N(f, g, h) = 10.$$

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In various special cases, in appropriate codimensional settings, this gives:

- Nielsen fixed point theory ($B = \Delta$)
- root theory ($B = pt$)
- coincidence theory of k maps ($B = \Delta \subset Y^k$)

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With apologies to Dobreńko and Kucharski, let's continue.

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Then F, G are maps of manifolds of the same dimension, and

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with $\mathrm{Coin}(F, G') = \mathrm{Eq}(f_1, f_2', \dots, f_k')$ finite.

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Easy to show that this is homotopy invariant, and has appropriate other properties.

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Equivalently, $x, x' \in \mathrm{Eq}(f_1, \dots, f_k)$ are in the same class when

$$x, x' \in p\,\mathrm{Eq}(\widetilde{f}_1, \alpha_2 \widetilde{f}_2, \dots, \alpha_k \widetilde{f}_k)$$

for $\alpha_i \in \pi_1(Y)$.

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We say $(\alpha_2, \dots \alpha_k) \sim (\beta_2, \dots, \beta_k)$ if and only if there is $z \in \pi_1(X)$ with

$$\beta_i = \varphi_1(z)\alpha_i\varphi_i(z)^{-1} \text{ for all } i$$

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$x, x' \in \mathrm{Eq}(f_1, \dots, f_k)$ are in the same class when there is a path γ from x to x' with

$$f_i(\gamma) \simeq f_1(\gamma) \text{ for all } i$$

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For more than 2 maps, this always holds except 3 maps on dimensions $2 \to 1$.

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Theorem

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If $f_1, \dots, f_k : T^{(k-1)n} \to T^n$ by matrices $A_1, \dots, A_k$, then

$$N(f_1, \dots, f_k) = \operatorname{abs}\det \begin{bmatrix} A_1 - A_2 \\ \vdots \\ A_1 - A_k \end{bmatrix}$$

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Any coincidence class containing an essential equalizer class must be geometrically essential.

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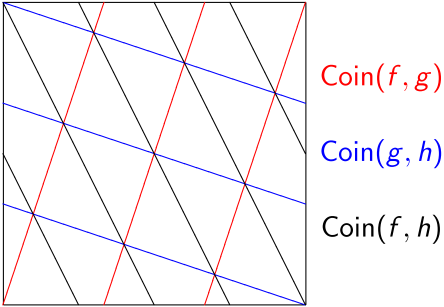

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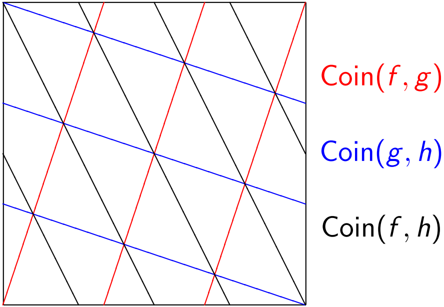

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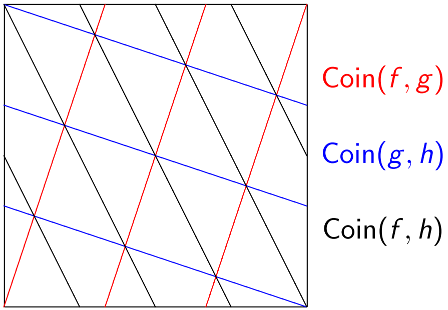

$\mathrm{Coin}(f, h)$ has 2 components which are 2 different classes. Are these classes geometrically essential?

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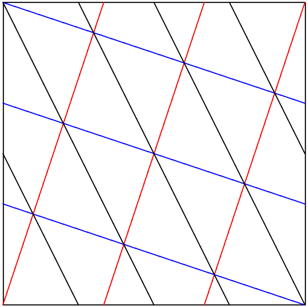

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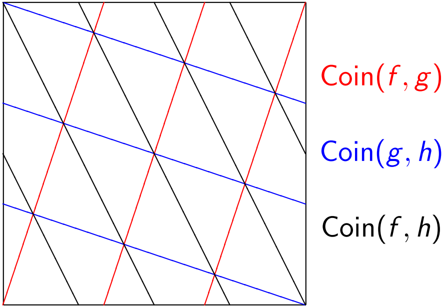

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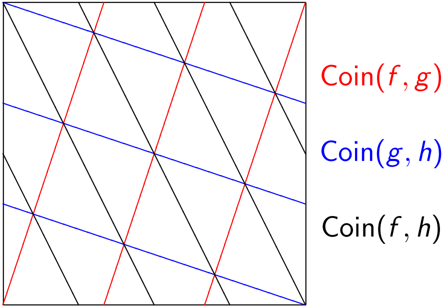

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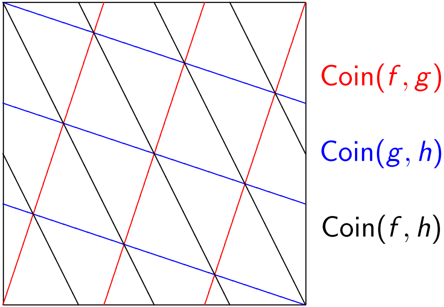

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So Jezierski *decreases* the domain dimension to get codimension 0.

But even when $\dim X$ isn't a multiple of $\dim Y$, maybe we can still make it work.

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Jezierski, *The Nielsen coincidence number of maps into tori*, *Quaestiones Mathematicae*, 2001
gives a method for finding $N(f_1, f_2)$ by observing that they restrict to maps $T^2 \to T^2$, and this restriction respects the Nielsen number.

So Jezierski *decreases* the domain dimension to get codimension 0.

This only works because $T^2 \subset T^7$.

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Let B_1, B_2 be matrices of $\overline{f}_1, \overline{f}_2$, and $B_2 - B_1$ still has rank 2.

So we can invent matrices $B_3, \dots B_5$ with

$$\begin{bmatrix} B_2 - B_1 \\ \vdots \\ B_5 - B_1 \end{bmatrix}$$

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Hopefully this trick can be used elsewhere when we need to prove that coincidence classes are essential.

Thank you!

Paper at arxiv: "Nielsen Equalizer Theory", and in *Topology and its applications*