Generic doubly-twisted conjugacy classes in free groups (preliminary)

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Setting

- G and H are finitely generated free groups, rank $H > 1$
- $\varphi, \psi : G \to H$ are homomorphisms
- Elements $u, v \in H$ are *doubly-twisted conjugate* iff there is some $g \in G$ with
$$u = \varphi(g) v \psi(g)^{-1}.$$
- No algorithm exists to decide doubly-twisted conjugacy in free groups.
- $[u]$ denotes the doubly-twisted conjugacy class of u.

Motivation: Nielsen coincidence theory

If $f, g : X \to Y$ are maps on compact surfaces with boundary, their induced homomorphisms in π_1 are as in the setting above.
The *coincidence set*:

$$\mathrm{Coin}(f, g) = \{x \in X \mid f(x) = g(x)\}$$

- Each coincidence x has an associated doubly-twisted conjugacy class (the *Nielsen class*)
- Two coincidences can be merged by a homotopy only if their Nielsen classes are the same.
- The *Nielsen number* $N(f, g)$ is the number of "essential" Nielsen classes, and is a lower bound on the minimal number of coincidences when f and g are changed by homotopies.
- Computing $N(f, g)$ typically requires deciding doubly-twisted conjugacy relations.

Homomorphisms with remnant

Let $G = \langle g_1, \ldots, g_n \rangle$. We say that φ *has remnant* if for every i, the word $\varphi(g_i)$ has a subword which does not cancel in any product like

$$\varphi(g_j)^{\pm 1}\varphi(g_i) \quad \text{or} \quad \varphi(g_i)\varphi(g_j)^{\pm 1}$$

except $j = i$ with exponent -1.

This is close to saying that $\{\varphi(g_1), \ldots, \varphi(g_n)\}$ is Nielsen reduced.

Very easy to check.

Remnants

I said it was easy

$$\varphi : \begin{array}{ccc} a & \mapsto & a^3bab^{-2} \\ b & \mapsto & ba^4ba^{-2} \end{array}$$

This one *has* remnant.

The condition is easy to check, and is generically satisfied for long words. (In fact for long words, we expect the remnant itself to be long.)

Remnants and doubly twisted conjugacy classes

Let φ^v be φ, conjugated by v.
Let $*$ be the free product, and $\varphi^v * \psi : G * G \to H$ the obvious homomorphism.

Theorem

Let $u \neq v$. If $\varphi^v * \psi$ has remnant, and the remnants of $\varphi^v(g_i)$ and $\psi(g_i)$ do not cancel in any product:

$$\varphi^v(g_i)v^{-1}u, \quad \psi(g_i)v^{-1}u, \quad u^{-1}v\varphi^v(g_i), \quad u^{-1}v\psi(g_i)$$

then

$$[u] \neq [v].$$

Note that these hypotheses are all likely to hold when φ and ψ are "long".

Density of non-conjugate pairs

Theorem

If φ and ψ have remnant, then random elements u, v will have $[u] \neq [v]$ with probability

$$P \geq 1 - \frac{1}{e(r)}$$

where $e(r)$ is exponential in the remnant lengths of φ and ψ.

In fact, any random finite set of elements will have pairwise disjoint twisted conjugacy classes with probability close to 1 as φ and ψ get long.

Back to Nielsen theory

This suggests a new approach to an old problem in Nielsen theory of compact manifolds:
Let $N(f,g)$ be the Nielsen number, $MC(f,g)$ the minimal coincidence number.

- From definitions, $N(f,g) \leq MC(f,g)$.
- (Wecken 1942, Schirmer, 1955) In dimension not equal to 2, $N(f,g) = MC(f,g)$.
- (Jiang, 1984) An example where $N(f,g) \neq MC(f,g)$ in dimension 2 (a surface with boundary).

Old question: What conditions on f, g will guarantee $N(f,g) = MC(f,g)$? Are these conditions commonly satisfied?

(Jiang, Guo, 1993): $N(f, \mathrm{id}) = MC(f, \mathrm{id})$ for homeomorphisms.

A conjecture

We conjecture that $N(f,g) = MC(f,g)$ generically for surfaces with boundary.

Random group elements are typically in different classes, and so coincidences with random Nielsen classes typically cannot be combined by homotopy.

So for random data, the number of Nielsen classes will be the same as the number of coincidences. (But how random is it really?)

Self-promotion

The paper (soon) and an online twisted-conjugacy calculator are at

- http://www.messiah.edu/~cstaecker