Dynamics of random selfmaps of surfaces with boundary and graphs

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Our theorem really is that for "almost all" maps f, the sequence $\{N(f^n)\}$ is nonzero and grows exponentially in n.

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There are only finitely many homomorphisms of each length.

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Imagine that your "randomly chosen homomorphism" will always be very long.

So when I say "almost all maps have property $\mathcal{P}$" or "a random map has $\mathcal{P}$ with probability 1" I mean that $D(S) = 1$ for the set of homotopy classes of maps with $\mathcal{P}$.

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Here, $\pi_1(X) = \langle a, b \rangle$ is a free group on two generators, and $f_\# : \pi_1(X) \to \pi_1(X)$ looks something like:

$$f_\# : \begin{array}{ccc} a & \mapsto & abab^2 \\ b & \mapsto & b^2a^{-1}ba^3 \end{array}$$

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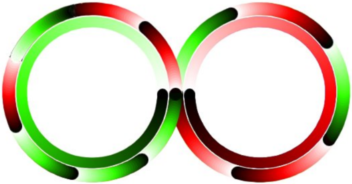
I imagine this map like this picture (*a* on the right, *b* on the left):



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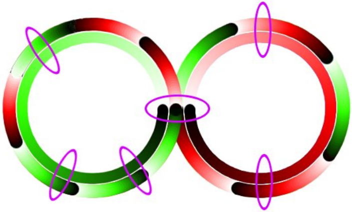
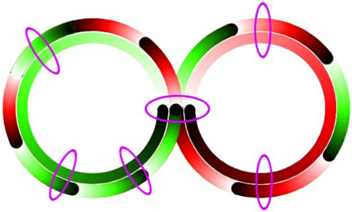
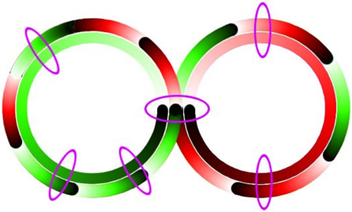


we can see the fixed points.



We get a fixed point every time $c^{\pm 1}$ appears inside the word $f(c)$.



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How common is this for a "random" map f?

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Strange things could happen when you iterate though. Letters giving fixed points could cancel after iteration, perhaps resulting in fewer fixed points for f^2 than for f.

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For a homomorphism $f_\#$ on the group $\langle a_1, \dots a_n \rangle$, we say $f_\#$ *has remnant* when there are subwords of each $f_\#(a_i)$ which never cancel in any product like

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When we iterate a map with remnant, it *grows*! The remnant subwords never cancel, and so just keep building up.

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So this map is in S_2:

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It's not too hard to show:

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If $f_\# \in S_k$, then

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$$\mathit{Growth}\{\#\,\mathrm{Fix}\, f^n\} = \lim_{n\to\infty} (\#\,\mathrm{Fix}\, f^n)^{1/n} > r$$

It takes more work, but the above arguments can be adapted using techniques by Hart, Heath, Keppelmann ('08) to hold for sets of minimal periodic points (not just $\operatorname{Fix} f^n$).

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But we show that for any r, almost all maps have $N^\infty(f) > r$.

So the growth of $\#\,\text{Fix}(f^n)$ for almost all f is exponential with arbitrarily high (but finite) growth rate.

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We show the same result for the fundamental group entropy $h_\#(f)$. (Though generally $h(f) \geq h_\#(f)$.)

Thank you!

Paper at arxiv: Kim, Staecker *Dynamics of random selfmaps of surfaces with boundary*

Or my website: `http://faculty.fairfield.edu/cstaecker`