# All kinds of big: Hadwiger's theorem

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Goal:

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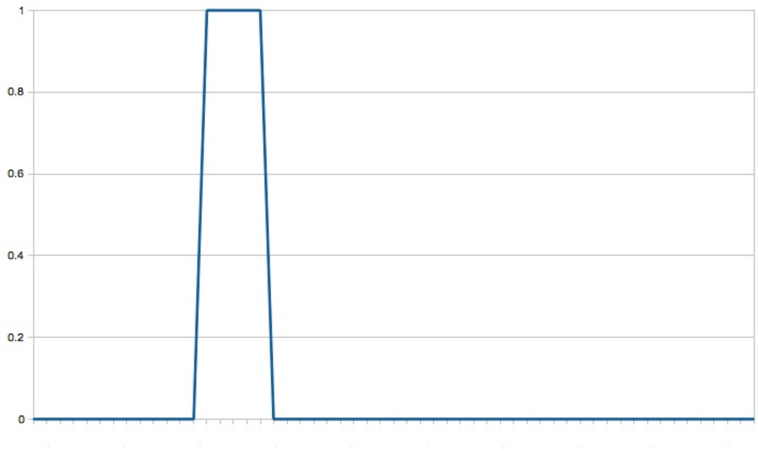
**Hadwiger's Theorem**:
If $v$ is a measure of bigness for sets in $\mathbb{R}^n$, then $v$ must have the form . . .



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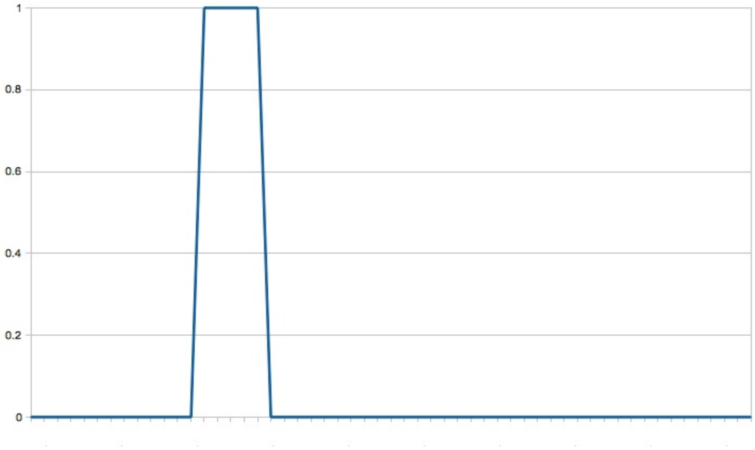


A graph of jokes per slide.

*every possible type of measure of "bigness" for subsets in space*

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“inclusion-exclusion”

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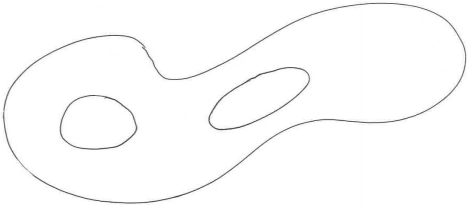

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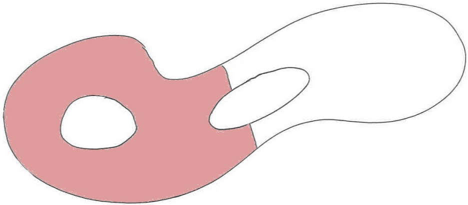

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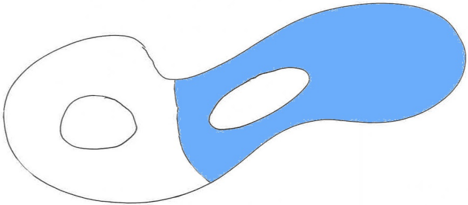

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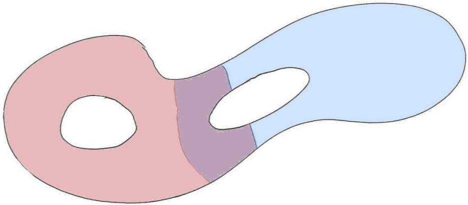

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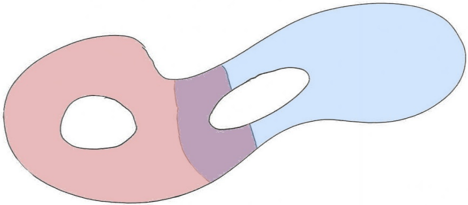

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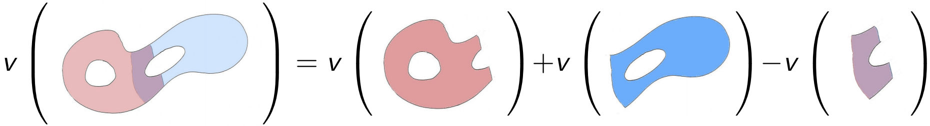
Then this says:



So by “measure of bigness” we technically mean:

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The area is one such function, but there are many others.

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Moral: volume and area of "pathological" sets don't add up the way we expect.

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Our continuity assumption is actually "continuity on convex sets"

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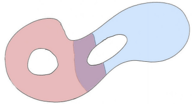
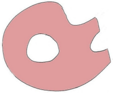
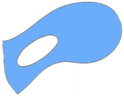
In $\mathbb{R}^2$, the area.

$v\Big($  $\Big) = v\Big($  $\Big) + v\Big($  $\Big) - v\Big($  $\Big)$

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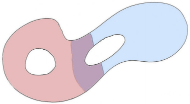
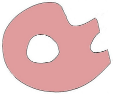
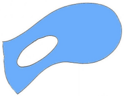
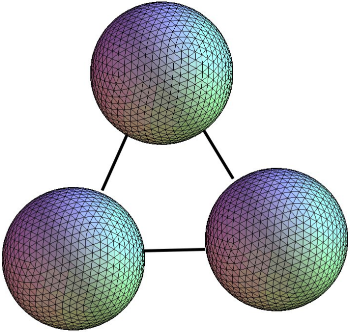
$v$ (  ) $= v$ (  ) $+ v$ (  ) $- v$ (  )

Also the perimeter!

In $\mathbb{R}^3$:



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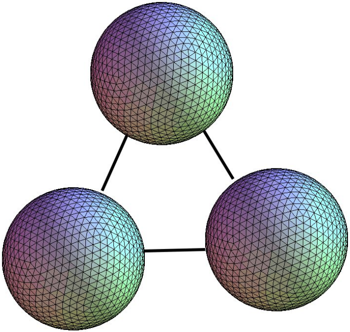

we have:

- the surface area

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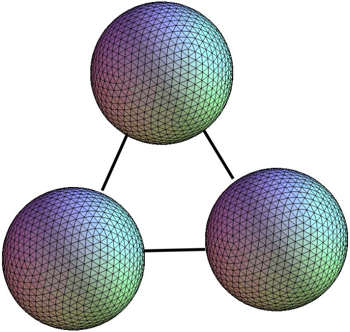

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- the surface area: $3 \cdot 4\pi = 12\pi$

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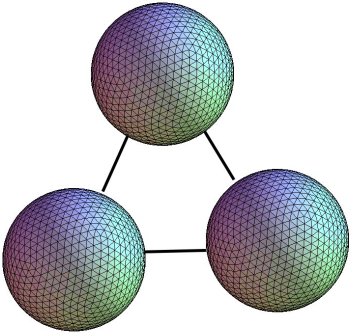

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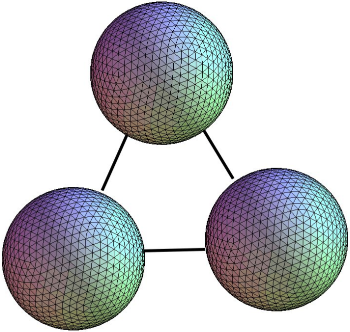

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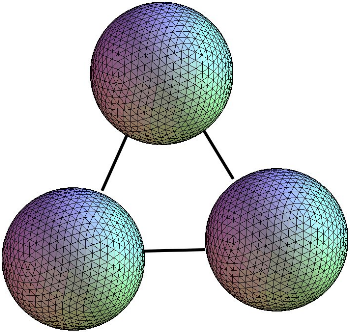

we have:

- the surface area: $3 \cdot 4\pi = 12\pi$
- the "perimeter": 3
- the volume

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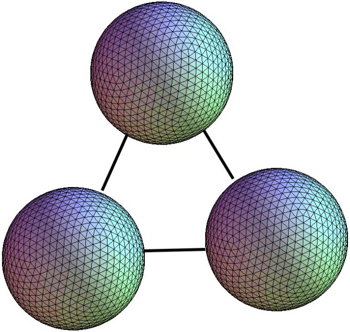

we have:

- the surface area: $3 \cdot 4\pi = 12\pi$
- the "perimeter": 3
- the volume: $3 \cdot (\frac{4}{3}\pi) = 4\pi$

- the volume:

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For higher dimensional spaces, there are higher dimensional intrinsic volumes.


1
0.8
0.6
0.4
0.2
0
You are here

The intrinsic volumes in each dimension are continuous invariant valuations.

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Are there any others?

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Are there any others?

Yes there are.

Our goal is to describe all possible continuous invariant valuations.

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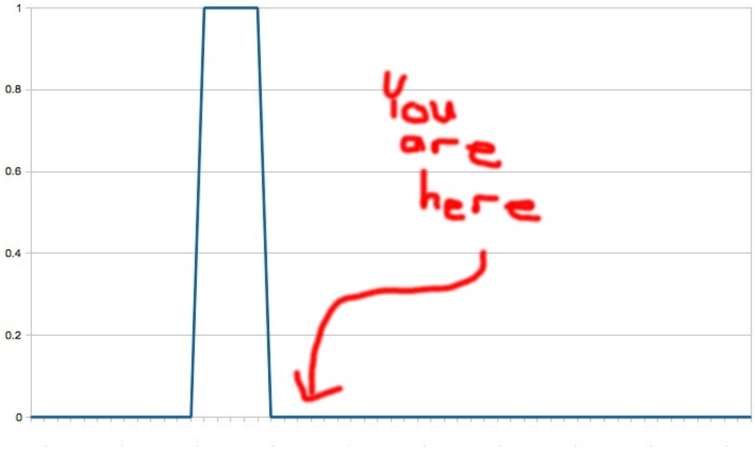
"All kinds of big"


HAVE YOU EVER HAD A REALLY BIG SECRET?
Tom Hanks
big


MR. BIG
To Be With You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Really stupid answer: zero

You could also do “perimeter plus area”

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Any sum of two continuous invariant valuations is a continuous invariant valuation.

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So the set of continuous invariant valuations is a vectorspace.

So there are infinitely many of them, but we can still try to find a basis for the space.

Other than the intrinsic volumes, are there any other really different continuous invariant valuations?

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There are!

Define a valuation $\chi$ like so:

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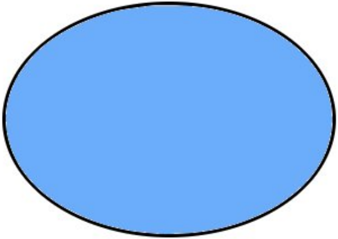
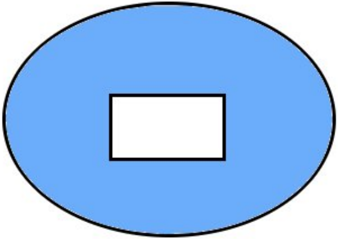
So for this:



we have $\chi(X) = 1$.

What about this:



Not convex, so break it up.

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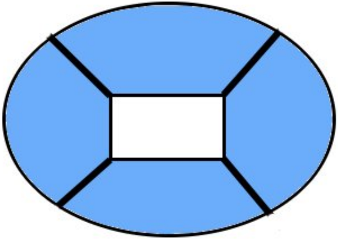

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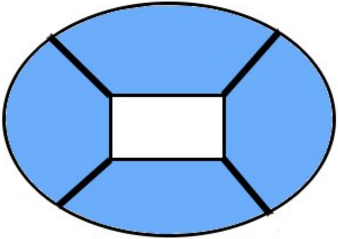

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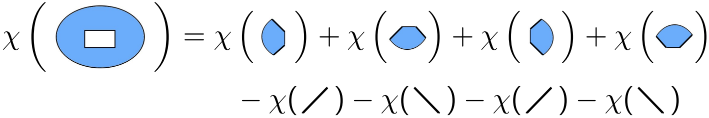

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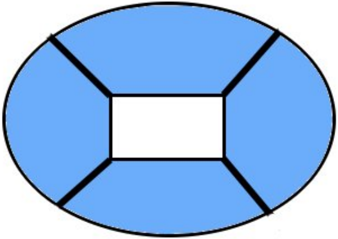

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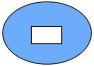
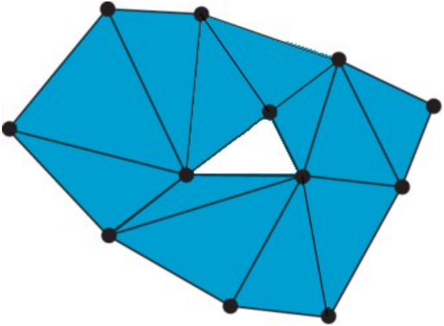
$\chi\Big($  $\Big) = \chi\Big($  $\Big) + \chi\Big($  $\Big) + \chi\Big($  $\Big) + \chi\Big($  $\Big)$

$$- \chi(\diagup) - \chi(\diagdown) - \chi(\diagup) - \chi(\diagdown)$$

$$= 1 + 1 + 1 + 1 - 1 - 1 - 1 - 1 = 0$$

We can do this computation in a more systematic way:



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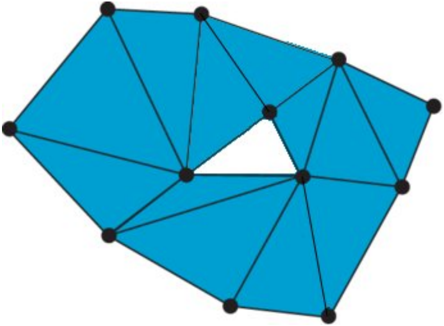

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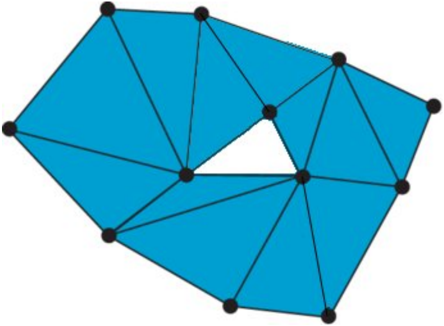

Make a triangulation. Decompose the space as faces, edges, vertices.

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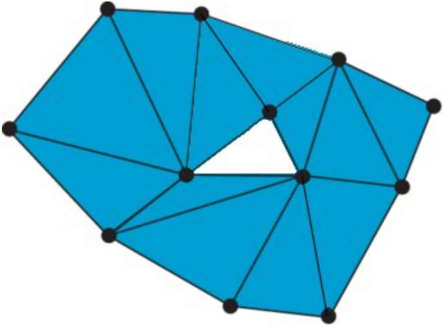

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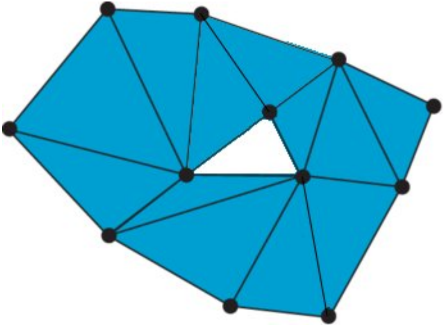

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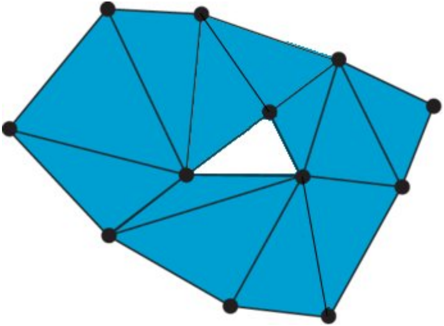

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Then $\chi$ is:

$$(\#\text{faces}) - (\#\text{edges}) + (\#\text{vertices})$$

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- any linear combination of these

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So any measure of bigness is some (unique) combination of intrinsic volumes and Euler characteristic.

In $\mathbb{R}^3$, this means that any measure of bigness has the specific form:

$$v(X) = c_0\chi(X) + c_1 P(X) + c_2 A(X) + c_3 V(X)$$

where $\chi$ is the Euler characteristic, $v_1$ is the perimeter, $v_2$ is the surface area, $v_3$ is the volume, and $c_i$ are constants.

This is actually a beautiful theorem.

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But it turns out that the valuation property is very restrictive.

The classification is much simpler than it should be.

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If your invariant is going to be a $\mathbb{R}$-valued valuation, it must be the Euler characteristic.

What remains:

- Why it's true
- Real-world applications

Let's try to give an idea of why any continuous invariant valuation must be some intrinsic volume.

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(Some familiar big ideas coming)

The intrinsic volumes break down nicely into dimensions.

Let's just show that the only "dimension 2" valuation in $\mathbb{R}^2$ is the area.

Specifically we'll show that the only (continuous invariant) valuation (which is zero on sets of dimension less than 2) is (a constant times) the area.

Let $v$ be any continuous invariant valuation in $\mathbb{R}^2$ which is zero on sets of dimension less than 2

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We'll show that $v(X) = c \cdot A(X)$ where $A$ is the area.

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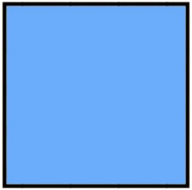
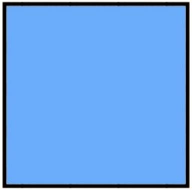
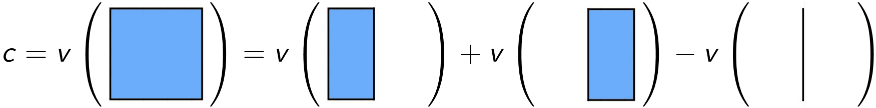
By invariance, any square of area 1 will have value $v(S) = c$.

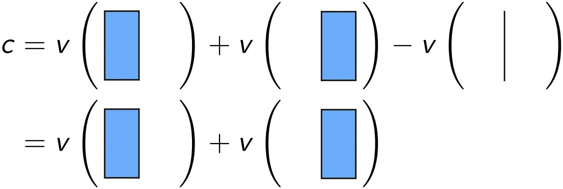


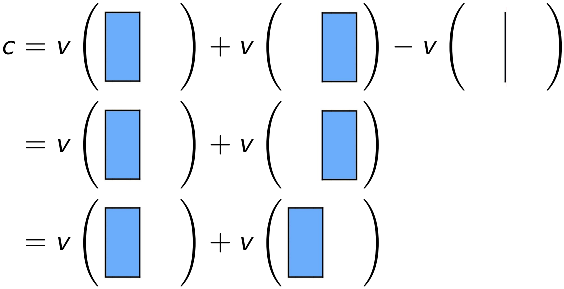


By the valuation property:

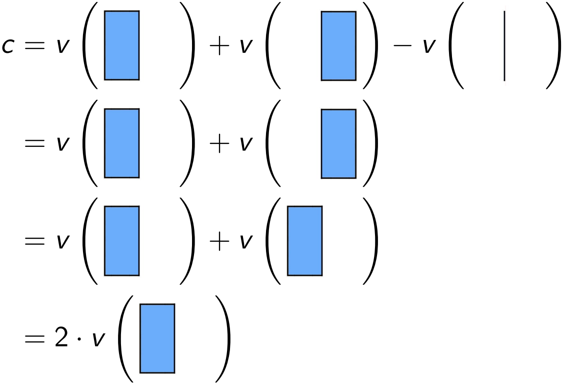


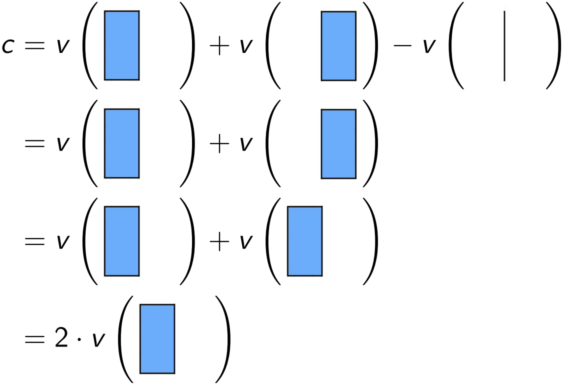
$c = v\Big($  $\Big) + v\Big($  $\Big) - v\Big($  $\Big)$

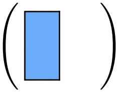




c = v
+ v
− v
= v
+ v
= v
+ v


c = v
+ v
− v
= v
+ v
= v
+ v
= 2 · v



So $v$  $= \frac{1}{2}c$.

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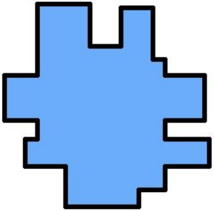
By cutting up different ways, easy to show that $v$ on a rectangle with area $q \in \mathbb{Q}$ is $c \cdot q$.

Already it's starting to look like $v$ is always just $c$ times the area, but we showed it only for rectangles.

What if our shape isn't a rectangle?



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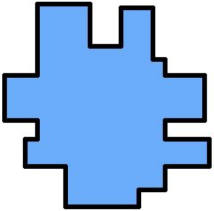

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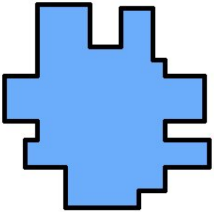

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For a shape like this, still $v$ must be $c$ times the actual area.

What if the shape is curvy?

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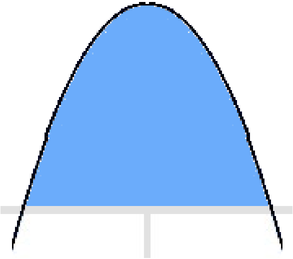

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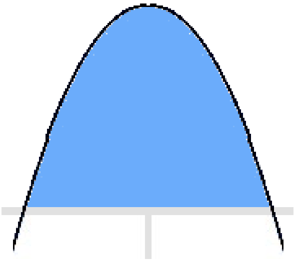
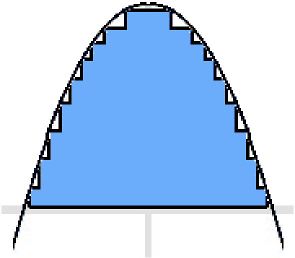


Cover it with rectangles!

# COVER IT WITH RECTANGLES!



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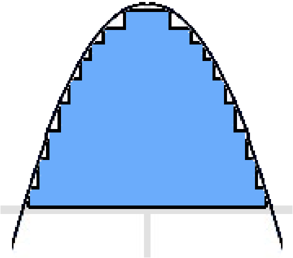

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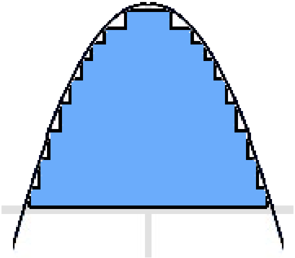

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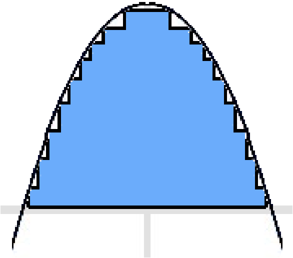

The area of the "rectified" region is close to the area of the curved region,

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Is the same true for $v$?

We already know $v$ is $c$ times the area for the rectified areas.

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It will because $v$ is continuous!

# The whole idea at once

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So any dimension 2 measurement which can be "broken down" additively must actually be the area (times a constant).

# Real-world applications

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What I'm about to say is mostly true.

Some examples are collected in: “Additivity, Convexity, and Beyond: Applications of Minkowski Functionals in Statistical Physics” by Mecke (2000)

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Given a flexible flat membrane (zero or uniform thickness), how much energy is required to bend it?

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Let’s ignore all that- assume constant temperature, etc. We care only about the shape of it.

What could the curvature energy depend on? (in terms of the shape)

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Actually we have no idea.

Beyond just the area, it probably depends somehow on the shape.

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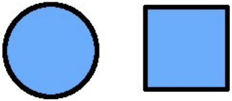
Specifically: Is the curvature energy the same for these?



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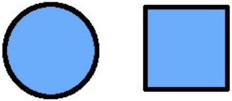
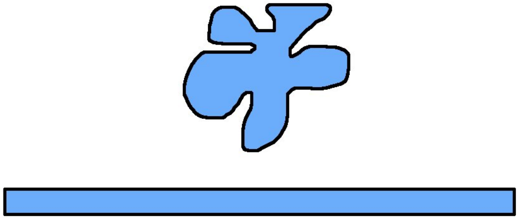


They have the same area, but they're different shapes.

# How about these?



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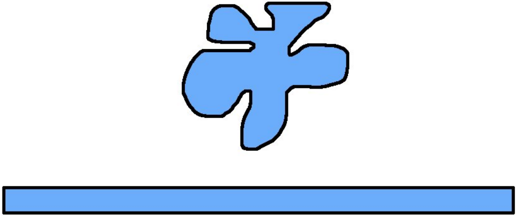

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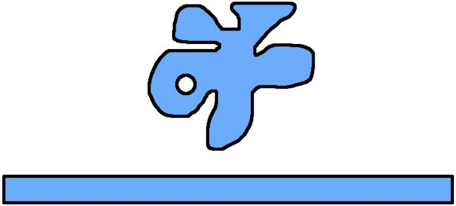

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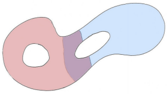
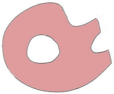
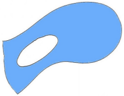
But it turns out the curvature energy is a valuation:

$E\Bigg($  $\Bigg) = E\Bigg($  $\Bigg) + E\Bigg($  $\Bigg) - E\Bigg($  $\Bigg)$

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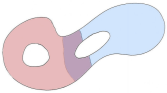
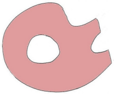
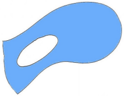
$E\Bigg($  $\Bigg) = E\Bigg($  $\Bigg) + E\Bigg($  $\Bigg) - E\Bigg($  $\Bigg)$

It is obviously continuous and invariant.

So by Hadwiger's theorem the curvature energy must have this form:

$$E(X) = c_1\chi(X) + c_2P(X) + c_3A(x)$$

where $\chi$ is the Euler characteristic, $P$ is the perimeter, and $A$ is the area.

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The only things we need to test experimentally are the constants.

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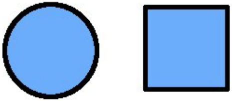

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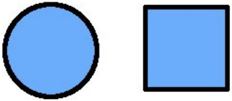

No- different perimeters.

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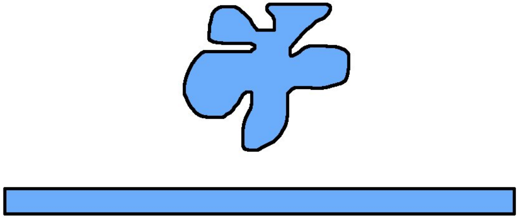

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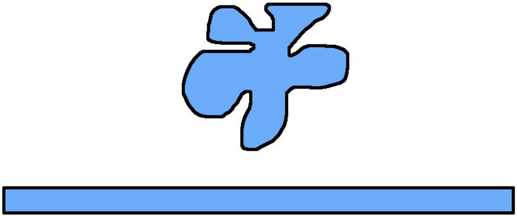

Yes- same areas, same perimeters, same Euler characteristic.

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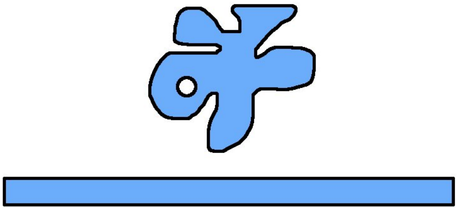

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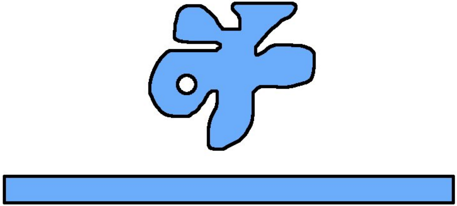

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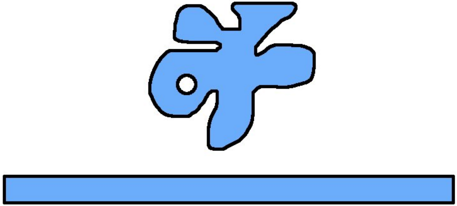

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(top is 0, bottom is 1)

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# How I came to this

I'm interested in the Euler characteristic, and there is another theorem by Watts, which looks just like Hadwiger's theorem in dimension 0.

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*(Hadwiger) The Euler characteristic $\chi$ is the unique function with:*

- $\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$
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## Theorem

*(Watts, 1962) The "reduced Euler characteristic" $\overline{\chi} = \chi - 1$ is the unique function with:*

- *When $A \subseteq B$,* $\overline{\chi}(B) = \overline{\chi}(A) - \overline{\chi}(B/A)$

# How I came to this

I'm interested in the Euler characteristic, and there is another theorem by Watts, which looks just like Hadwiger's theorem in dimension 0.

### Theorem

*(Hadwiger) The Euler characteristic $\chi$ is the unique function with:*

- $\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$
- *When $X$ is convex,* $\chi(X) = 1$

### Theorem

*(Watts, 1962) The "reduced Euler characteristic" $\overline{\chi} = \chi - 1$ is the unique function with:*

- *When* $A \subseteq B$, $\overline{\chi}(B) = \overline{\chi}(A) - \overline{\chi}(B/A)$
- $\overline{\chi}(S^0) = 1$

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A major tool is the Lefschetz number $L(f)$ of a map from a space to itself.

Always $L(\mathrm{id}) = \chi(X)$, so $L(f)$ is a generalization of the Euler characteristic.

Think of $L(f)$ like an Euler characteristic for a function.

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I did some stuff with this too.

So when I saw Hadwiger's theorem, I knew immediately that it would give yet another theorem about $L(f)$.

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### Theorem

*There is a unique function $\Lambda : N(X) \to \mathbb{R}$ satisfying:*

- *Let $A, B$ be subcomplexes of some common subdivision of $X$. Then $\Lambda(f, \emptyset) = 0$, and*

$$\Lambda(f, A \cup B) = \Lambda(f, A) + \Lambda(f, B) - \Lambda(f, A \cap B).$$

- *Let $f$ be a Hopf simplicial map and $x$ be a simplex. If $x$ is not a maximal simplex we have $\Lambda(f, x) = 0$, and if $x$ is a maximal simplex we have*

$$\Lambda(f, x) = (-1)^{\dim X} c(f, x).$$

- *$\Lambda(f, A)$ depends continuously on $f$.*

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Currently looking at higher dimensions.

That's all!