The uniqueness of the coincidence index on orientable differentiable manifolds

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Setting

Throughout, we study continuous maps $f, g : X \to Y$, where X and Y are connected, orientable, differentiable manifolds of the same dimension n.

For some $U \subset X$,

$$\text{Coin}(f, g, U) = \{x \in U \mid f(x) = g(x)\}.$$

The *coincidence index* assigns an integer value

$$\text{ind}(f, g, U) \in \mathbb{Z},$$

a sort of multiplicity count for coincidence points which sums to the Lefschetz number and aids in the computation of the Nielsen number.

Axioms in fixed point theory

The fixed point index ind(f, U) for selfmaps is defined on ANRs. A uniqueness result for polyhedra using 5 axioms was given by O'Neill, 1953.

In 2004, a uniqueness result for (possibly nonorientable) manifolds: Let $\mathcal{C}$ be the set of *admissable* pairs (f, U) where $f : X \to X$ is a selfmap of a differential manifold, and $U \subset X$ is open with Fix(f, U) compact.

Theorem (Furi, Pera, Spadini, 2004). *There is a unique function*

$$\text{ind} : \mathcal{C} \to \mathbb{R}$$

satisfying the following three axioms:

- **(Normalization)** *If* $c : X \to X$ *is constant, then*

$$\text{ind}(c, U) = 1.$$

- **(Homotopy)** *If* f *is admissably homotopic to* f', *then*

$$\text{ind}(f, U) = \text{ind}(f', U).$$

- **(Additivity)** *If* $\text{Fix}(f, U) \subset A \sqcup B$, *then*

$$\text{ind}(f, U) = \text{ind}(f, A) + \text{ind}(f, B).$$

Axioms for the coincidence index

For X, Y orientable differential manifolds of dimension n, let $\mathcal{C}(X, Y)$ be all *admissable triples* (f, g, U) for $f, g : X \to Y$.

Theorem. *There is a unique function*

$$\text{ind} : \mathcal{C}(X, Y) \to \mathbb{R}$$

satisfying three axioms:

Homotopy Axiom. If (f, g) is admissably homotopic to (f', g'), then

$$\text{ind}(f, g, U) = \text{ind}(f', g', U).$$

Additivity Axiom. If $\text{Coin}(f, g, U) \subset A \sqcup B$, then

$$\text{ind}(f, g, U) = \text{ind}(f, g, A) + \text{ind}(f, g, B).$$

The Normalization axiom

Recall the fixed-point normalization axiom:

$$\text{ind}_{\mathsf{Fix}}(c, X) = 1$$

This is difficult to generalize to coincidence theory. A naïve attempt will try:

$$\text{ind}(c, \text{id}, X) = 1$$

This isn't quite strong enough (though it will work if we allow only selfmaps $f, g : X \to X$).

Normalization Axiom. If $L(f, g)$ is the coincidence Lefschetz number, then

$$\text{ind}(f, g, X) = L(f, g)$$

Straight from the axioms

Proposition 1 (Weak normalization). *If* $c : X \to X$ *is constant and* id : $X \to X$ *is identity, then*

$$\text{ind}(c, \text{id}, X) = 1.$$

Proposition 2 (Fixed point index). *For a selfmap* $f : X \to X$*,*

$$\text{ind}_{\text{Fix}}(f, U) = \text{ind}_{\text{Coin}}(f, \text{id}, U)$$

Proposition 3 (Solution). *If* $\text{ind}(f, g, U) \neq 0$*, then* f *and* g *have a coincidence on* U*.*

Proof strategy

We prove the uniqueness in steps:

- Linear maps $\mathbb{R}^n \to \mathbb{R}^n$
- Nondegenerate maps $\mathbb{R}^n \to \mathbb{R}^n$
- Nondegenerate maps $X \to Y$
- All maps $X \to Y$

Linear maps $\mathbb{R}^n \to \mathbb{R}^n$

For linear maps $A, B : \mathbb{R}^n \to \mathbb{R}^n$, the coincidence set is easy:

$$\text{Coin}(A, B, \mathbb{R}^n) = \ker(B - A)$$

Triples like $(A, B, \mathbb{R}^n) \in \mathcal{C}(\mathbb{R}^n, \mathbb{R}^n)$ with A, B linear are just maps with $B - A \in Gl_n$.

We know Gl_n has two path components, and so by the homotopy axiom, the index is constant on each component. The weak normalization property gives

Lemma. If A, B are linear, then

$$\text{ind}(A, B, \mathbb{R}^n) = \text{sign}\det(B - A).$$

Nondegenerate maps $\mathbb{R}^n \to \mathbb{R}^n$

We say $(f, g, U) \in \mathcal{C}(X, Y)$ is *nondegenerate* if f, g are differentiable and

$$dg_x - df_x \in Gl_n$$

for every coincidence point $x \in U$.

Lemma. If $(f, g, U) \in \mathcal{C}(\mathbb{R}^n, \mathbb{R}^n)$, then each coincidence point is isolated, and for $V_x \subset U$ an isolating neighborhood of x, we have

$$\begin{aligned} \mathrm{ind}(f, g, V_x) &= \mathrm{ind}(df_x, dg_x, \mathbb{R}^n) \\ &= \mathrm{sign}\,\mathrm{det}(dg_x - df_x). \end{aligned}$$

The isolation is an analytical argument. The index is a fancy application of the homotopy axiom.

Nondegenerate maps $X \to Y$

For maps $f, g : X \to Y$ and a coincidence point x with $y = f(x) = g(x)$, take charts $\varphi_x : U_x \to \mathbb{R}^n$ and $\psi_y : W_y \to \mathbb{R}^n$. Then there is a correspondence:

$$\mathrm{Coin}(f, g, U_x) \leftrightarrow \mathrm{Coin}(\psi_y \circ f \circ \varphi_x^{-1}, \psi_y \circ g \circ \varphi_x^{-1}, \mathbb{R}^n)$$

We use this correspondence to leverage our earlier results on $\mathcal{C}(\mathbb{R}^n, \mathbb{R}^n)$.

Lemma. If (f, g, U) is nondegenerate, each coincidence point x is isolated, and

$$\mathrm{ind}(f, g, U) = \sum_{x \in \mathrm{Coin}(f,g,U)} \mathrm{sign}\det(dg_x - df_x).$$

The key step is to show that the correspondence via the charts preserves the index. This requires the full normalization axiom.

Maps $X \to Y$

Lemma. Every triple in $\mathcal{C}(X, Y)$ is admissably homotopic to a nondegenerate triple

Now to compute ind(f, g, U), first replace (f, g, U) by a nondegenerate triple. Then the index is simply the sum of the signs of determinants of derivatives.

By the homotopy axiom, the above is independant of our choice of nondegenerate triple, and so uniquely defines the index.