Axioms for the Lefschetz number as a lattice valuation

P. Christopher Staecker

Fairfield University, Fairfield CT

Nielsen Theory and Related Topics 2013

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Based on axioms for $\chi(X)$ by Watts.

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- *(Wedge of circles) If f is a map on a wedge of k circles, then*
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- *(Commutativity) $\overline{L}(f \circ g) = \overline{L}(g \circ f)$*

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$$\mathrm{ind}(c, X) = 1$$

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Some also have the commutativity property.

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A&B seems hard to extend to coincidence theory,
FP&S seems hard to do for nonmanifolds. (without commutativity)

Hadwiger's Theorem

Our scheme for $L(f)$ is based on Hadwiger's Theorem (1950s), for subcomplexes of an abstract simplicial complex:

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- *(Valuation axiom)* $\chi(\emptyset) = 0$ *and if* A, B *are subcomplexes of* X, *then*

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- *(Simplex axiom) If x is a simplex, then $\chi(x) = 1$.*

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Two operations $\cap$ and $\cup$ which are commutative, associative, distributive, with a few more properties.

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We'll use the same lemma for our result.

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Theorem

There is a unique function $L : M(X) \to \mathbb{R}$ satisfying:

- *(Valuation axiom) $L(f, \emptyset) = 0$ and if A, B are subcomplexes of X, then*

$$L(f, A \cup B) = L(f, A) + L(f, B) - L(f, A \cap B)$$

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- *(Simplex axiom) If x is a simplex, then and*

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If $f(x) = x$ then $c(f, x) = \pm 1$ depending on the orientation.

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Adding these up, it's easy to verify that

$$L(f, X) = \sum_q (-1)^q \operatorname{tr}(f_q : C_q(X) \to C_q(X)),$$

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Note: we obtain this trace formula even without assuming a homotopy invariance axiom.

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So we need a “subdivision” version of the theorem.

Let $M'(X)$ be the set of pairs (f, A), where A is a subcomplex of some subdivision X' of X, and $f : X' \to X$ is simplicial.

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Theorem

There is a unique function $L : M'(X) \to \mathbb{R}$ satisfying:

- *(Valuation axiom) $L(f, \emptyset) = 0$ and if A, B are subcomplexes of a common subdivision of X, then*

$$L(f, A \cup B) = L(f, A) + L(f, B) - L(f, A \cap B)$$

- *(Simplex axiom) If x is a simplex, then*

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Let X be a compact polyhedron, and let $N(X)$ be the set of pairs (f, A) where $f : X \to X$ is continuous and A is a subpolyhedron of some subdivision of X.

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Let X be a compact polyhedron, and let $N(X)$ be the set of pairs (f, A) where $f : X \to X$ is continuous and A is a subpolyhedron of some subdivision of X.

Then our previous arguments suffice in this setting, using a homotopy property to get simplicial approximations.

Theorem

There is a unique function $\Lambda : N(X) \to \mathbb{R}$ *satisfying:*

- *(Homotopy axiom) If* $f \simeq g$*, then* $\Lambda(f, A) = \Lambda(g, A)$*.*

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$$\Lambda(f, A \cup B) = \Lambda(f, A) + \Lambda(f, B) - \Lambda(f, A \cap B)$$

- *(Simplicial map axiom) If* f *is simplicial and* x *is a simplex, then*

$$\Lambda(f, x) = (-1)^{\dim x} c(f, x) + \Lambda(f, \partial x).$$

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Previous theorem gets the uniqueness

Need to check that alternative homotopies don't change the value, but we already have the trace formula which is homotopy invariant.

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A Hopf simplicial map has no fixed points on the boundaries, so in

$$\Lambda(f,x) = (-1)^{\dim x} c(f,x) + \Lambda(f,\partial x),$$

we'll always have $\Lambda(f,\partial x) = 0$.

So we get a weaker simplicial map axiom:

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$$\Lambda(f, A \cup B) = \Lambda(f, A) + \Lambda(f, B) - \Lambda(f, A \cap B)$$

- *(Hopf simplicial map axiom) Let f be Hopf simplicial. If x is a nonmaximal simplex then $L(f, x) = 0$. If x is a maximal simplex, then*

$$\Lambda(f, x) = (-1)^{\dim X} c(f, x).$$

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But the simplicial approximation theorem and Hopf construction require only small homotopies.

Actually the set of Hopf simplicial maps with fixed points in maximal simplices is a dense set in X^X, the space of selfmaps.

Since the valuation and simplicial map axioms determine Λ on a dense set, we need only assume continuity of Λ to have uniqueness on all of X^X.

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A "continuity axiom" is weaker.

So our final result is:

Theorem

There is a unique function $\Lambda : N(X) \to \mathbb{R}$ satisfying:

- *(Continuity axiom) The value $\Lambda(f, A)$ depends continuously on $f \in X^X$.*
- *(Valuation axiom) If A, B are subpolyhedra of a common subdivision of X, then $\Lambda(f, \emptyset) = 0$ and*

$$\Lambda(f, A \cup B) = \Lambda(f, A) + \Lambda(f, B) - \Lambda(f, A \cap B)$$

- *(Hopf simplicial map axiom) Let f be Hopf simplicial. If x is a nonmaximal simplex then $L(f, x) = 0$. If x is a maximal simplex, then*

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By the way, a similar weakening may be possible in the FPS approach.

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Conjecture

The fixed point index is the unique $\mathbb{R}$*-valued function satisfying the following axioms:*

- *(Continuity)* $\text{ind}(f, U)$ *depends continuously on* $f \in X^X$
- *(Additivity) If* $\text{Fix}(f) \cap U \subset U_1 \sqcup U_2$*, then*

$$\text{ind}(f, U) = \text{ind}(f, U_1) + \text{ind}(f, U_2)$$

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Not sure if this will work for the A&B approach.

Thanks!