Typical elements in free groups are in different doubly-twisted conjugacy classes

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Setting

- G and H are finitely generated free groups, rank $H > 1$
- $\varphi, \psi : G \to H$ are homomorphisms
- Elements $u, v \in H$ are *doubly-twisted conjugate* iff there is some $g \in G$ with
$$u = \varphi(g) v \psi(g)^{-1}.$$
- No algorithm exists to decide doubly-twisted conjugacy in free groups.
- $[u]$ denotes the doubly-twisted conjugacy class of u.

Anecdotal evidence

I've done a lot of computer tests for finding twisted and doubly-twisted conjugacy classes. We generate homomorphisms and elements at random, and then test to see if the elements are twisted conjugate.

As it happens, usually they are not!

We aim to show that if φ,ψ, u, v are all chosen at random, then $[u] \neq [v]$ with probability 1.

Homomorphisms with remnant

Let $G = \langle g_1, \ldots, g_n \rangle$. We say that φ *has remnant* if for every i, the word $\varphi(g_i)$ has a subword which does not cancel in any product like

$$\varphi(g_j)^{\pm 1}\varphi(g_i) \quad \text{or} \quad \varphi(g_i)\varphi(g_j)^{\pm 1}$$

except $j = i$ with exponent -1.

This is close to saying that $\{\varphi(g_1), \ldots, \varphi(g_n)\}$ is Nielsen reduced.

Very easy to check.

Remnants

I said it was easy

$$\varphi : \begin{array}{rcl} a & \mapsto & a^3bab^{-2} \\ b & \mapsto & ba^4ba^{-2} \end{array}$$

This one has remnant, the remnants are underlined.

We write

$$\operatorname{Rem}_\varphi a = bab^{-1}$$
$$\operatorname{Rem}_\varphi b = a^4b$$

It is easy to check if φ has remnant, and this is generically true for long words. (In fact for long words, we expect the remnant itself to be long.)

Remnant numbers

Some measures of the size of the remnant:

For a natural number ℓ, if $|\operatorname{Rem}_\varphi g| \geq \ell$ for all generators g, then we say that φ has *remnant length* ℓ.

For a real number $r \in (0,1)$, if $|\operatorname{Rem}_\varphi g| \geq r|\varphi(g)|$ for all generators g, then we say that φ has *remnant ratio* r.

So for

$$\varphi : \begin{array}{ccc} a & \mapsto & a^2 \underline{bab^{-1}} b^{-1} \\ b & \mapsto & b\underline{a^4 b}a^{-2} \end{array}$$

The remnant length is 3, and the remnant ratio is $1/2$.

Our coincidence remnant condition

We will have two homomorphisms φ, ψ. We will require not only that φ and ψ each have remnant, but that they have remnant when considered *together*.

For example:

$$\varphi : \begin{array}{ccc} a & \mapsto & a^2bab^{-2} \\ b & \mapsto & ba^4ba^{-2} \end{array}$$

$$\psi : \begin{array}{ccc} a & \mapsto & b^{-1}a^4 \\ b & \mapsto & a^2ba \end{array}$$

For our purposes, this is *not* good enough because $\psi(b)$ has too much cancellation with $\varphi(a^{-1})$.

The free product homomorphism

Consider the free product $G * G$, and there is a natural homomorphism $\varphi * \psi : G * G \to H$.

For $G = \langle g_i \rangle$, write $G * G = \langle g_i, g_i' \rangle$ and define

$$\begin{aligned} \varphi * \psi(g_i) &= \varphi(g_i) \\ \varphi * \psi(g_i') &= \psi(g_i) \end{aligned}$$

So from above,

$$\varphi * \psi : \begin{array}{ccc} a & \mapsto & a^2bab^{-2} \\ b & \mapsto & ba^4ba^{-2} \\ a' & \mapsto & b^{-1}a^4 \\ b' & \mapsto & a^2ba \end{array}$$

and we will require that $\varphi * \psi$ has remnant. (Here, it does not.)

An easy lemma

Lemma

If $\varphi * \psi : G * G \to H$ *has remnant, then* $\varphi(G) \cap \psi(G) = \{1\}$*.*

Proof.

If not, then there are nontrivial $x, y \in G$ with

$$\varphi(x) = \psi(y)$$

and so

$$\varphi(x)\psi(y)^{-1} = 1.$$

But this is impossible since $\varphi * \psi$ has remnant. If we write x and y in generators, the word cannot cancel. □

Moral: The condition that $\varphi * \psi$ has remnant is very strong.

Remnants and doubly twisted conjugacy classes

Let φ^v be φ, conjugated by v.

Theorem

Let $u \neq v$. If $\varphi^v * \psi$ has remnant, and if for any generator g the remnants $\text{Rem}_{\varphi^v * \psi}\, g$ *do not cancel in any product:*

$$(\varphi^v * \psi(g))v^{-1}u, \quad u^{-1}v(\varphi^v * \psi(g))$$

then

$$[u] \neq [v].$$

Note: the remnant hypothesis is never true for $\psi = \text{id}$.

But it will be true for “most” homomorphisms and “most” choices of u and v.

"Most" homomorphisms have remnant

Everybody knows Bob's theorem:

Theorem

(R. F. Brown (in Wagner), 1999) Given any $\varepsilon > 0$*, there is some* M *such that if* φ *is a random homomorphism with word lengths at most* M*, then* φ *has remnant with probability greater than* $1 - \varepsilon$*.*

This fits nicely into the theory of *generic* sets in groups, measured by *asymptotic density*.

Measuring the density is a great way to get a handle on properties which we want to show hold for "most" things.

Density and generic sets

Let H_p be all words in H of length at most p.

For a subset $S \subset H$, the *[asymptotic] density* of S is:

$$D(S) = \lim_{p \to \infty} \frac{S \cap H_p}{H_p}$$

If $D(S) = 1$, we say S is *generic*.

Similarly we can define the density of sets of tuples, and thus the density of sets of homomorphisms.
So Bob's theorem can be restated as: The set of homomorphisms $\varphi : G \to G$ with remnant is generic.

Property $C'(\lambda)$

Much stronger results are known about this kind of thing:
A subset S in a free group has property $C'(\lambda)$ when: given an element $x \in S$, the pieces of x cancelling in products with other elements of S (or their inverses) have length less than $\lambda|x|$.

Theorem

(Arzhantseva, Ol'shanskii, 1996) If H has rank greater than 1, then the set of subsets of H having small cancellation property $C'(\lambda)$ is generic for any $\lambda > 0$.

So A&O's theorem says that, when we look at the image words of a homomorphism, the remnant subwords will generically be arbitrarily big.

Fancier generic remnant properties

So Bob's theorem can be substantially improved:

Theorem

Let H have rank greater than 1. Then:

- *The set of homomorphisms $\varphi : G \to H$ with remnant is generic.*
- *The set of homomorphisms $\varphi : G \to H$ with remnant length at least ℓ is generic for any ℓ.*
- *The set of homomorphisms $\varphi : G \to H$ with remnant ratio r is generic for any $r \in (0, 1)$.*

Applied to homomorphisms $G * G \to H$, this means that $\varphi * \psi$ will generically have remnant subwords as long as we like.
In particular this means that $\varphi(G) \cap \psi(G)$ is generically trivial, which is a bit surprising. (Think about $\varphi, \psi : F_{1000} \to F_2$)

Main asymptotic result

Theorem

Let H have rank greater than 1, $\varphi, \psi : G \to H$ each have remnant ratio at least 1/2, and let k be the minimum length of any word $\varphi(g)$ or $\psi(g)$ for a generator g.

Generically, the remnant ratio is as big as we want, and k is also as big as we want.

Let S be the set of pairs (u, v) with $[u] \neq [v]$. Then

$$\lim_{k \to \infty} D(S) = 1.$$

We say S is "almost generic in k".

Corollary

The set of tuples (φ, ψ, u, v) with $[u] \neq [v]$ is generic.

The density estimate

In case you're wondering, the proof shows that

$$D(S) \geq 1 - 2(2m-1)^{-\frac{k-1}{2}} - (32n)^2(2m-1)^{-(k-1)},$$

where n and m are the ranks of G and H.

This density goes to 1, but takes a while to get there.

For maps $F_2 \to F_2$, (so $n = m = 2$) the density estimate is actually negative until $k = 9$.

k	$D(S)$ estimate
9	.35
10	.78
11	.92
12	.97
13	.99

Some obvious questions

Let S be the set of $u, v \in H$ with $[u] \neq [v]$.
Theorem says $D(S) \to 1$ as $|\varphi| \to \infty$ and $|\psi| \to \infty$

1. Fixing a *particular* pair of homomorphisms φ, ψ, is $D(S) = 1$?
   Not for all pairs: if $\varphi = \text{id}$ and $\psi = 1$ then $[u] = [v]$ for all u and v.
2. Fixing ψ but allowing φ to vary, does $D(S) \to 1$ as $|\varphi| \to \infty$?
   I think I can show this when ψ has remnant, using different methods
3. For singly-twisted conjugacy, does $D(S) = 1$ as $|\varphi| \to \infty$?
   $2 \Rightarrow 3$, and id has remnant
4. Do similar results hold in surface groups?

Back to Nielsen theory

This suggests a new approach to asking about Wecken maps on compact surfaces with boundary:
A selfmap f is *Wecken* if $N(f) = MF(f)$. A pair (f, g) is Wecken if $N(f, g) = MC(f, g)$. We can say that a pair of homomorphisms (φ, ψ) are Wecken in the obvious way.

A necessary condition for coincidence points to be merged is that their Reidemeister classes are in the same doubly-twisted conjugacy classes.

All things being chosen randomly, these classes should never be equal.

Conjectures

Conjecture

If the rank of H is greater than one, the set of Wecken pairs $\varphi, \psi : G \to H$ is generic.

The same should be true in fixed point theory, but different methods would be needed.

This is maybe some way in the distance as of now, but partial results would be nice too. Even showing that the Wecken maps have nonzero density would (I believe) be new and interesting.

(We know homeomorphisms are Wecken, but these are still density 0.)

An easy one:
Don't steal this idea

Wagner's algorithm says that $[u] \neq [v]$ unless certain Wagner tails match up.

If the Wagner tails are all different, none of the fixed points can be merged, and thus the map is Wecken.

Estimating the density of Wecken maps in this way is just a combinatorial exercise: what is the probability that all of the Wagner tails are different?

Prophecy:

My prediction: Wagner's algorithm, plus basic combinatorics, can show that the Wecken maps have nonzero density, but fancier methods will be needed to show that the density is 1.

Anectodal evidence: generate 10,000 random homomorphisms $F_n \to F_n$ with maximum word length k, and see how many of these have all Wagner tails different.

Experimental results

Density of homomorphisms with all Wagner tails different

For maps $F_2 \to F_2$:

k	Density
1	0
2	.19
3	.20
4	.22
5	.21
6	.20
7	.21
8	.20

For maps $F_3 \to F_3$:

k	Density
1	0
2	.08
3	.10
4	.19
5	.22
6	.24
7	.25
8	.25

For maps $F_4 \to F_4$:

k	Density
1	0
2	.04
3	.06
4	.18
5	.25
6	.27
7	.27
8	.27

It's over!

The paper is at my website and at arXiv.

See also my web-based twisted conjugacy calculator "The Nielsen Theory Web Machine", by me and Chris Putnam.

Experiments were done in GAP, code is at my website.